GOVERNMENT ARTS AND SCIENCE COLLEG, KOVILPATTI – 628 503.
(AFFILIATED TO MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI)
DEPARTMENT OF MATHEMATICS
STUDY E - MATERIAL
CLASS : I M.SC (MATHEMATICS) SEM: I
SUBJECT : NUMERICAL ANALYSIS(PMAM15)

1.5 Paper 5: NUMERICAL ANALYSIS

Text Book: Numerical Methods, S. Arumugam and others, Scikech(2001).

Unit I: Interpolation : Newton's Interpolation Formula – Central difference Interpolation Lagrange's Interpolation formula – Divided differences - Newton's Divided differences formula – Inverse Interpolation – Hermit's Interpolating Polynomial.

Chapter 7: Sections 7.1 to 7.7.

Unit II: Numerical differentiation – Derivatives using Newton's forward, backward, central difference formulae

Chapter 8: Sections 8.1 to 8.3.

Unit III: Numerical Integration –Gaussian Quadrature formula –Numerical evaluation of double integrals.

Chapter 8: Sections 8.5 to 8.7.

Unit IV: Numerical solutions of ordinary differential equations – Taylor's series Method – Picard's Method – Euler's Method – Runge Kutta Method.

Chapter 10: Sections 10.1 to 10.4.

Unit V: Predictor corrector Method – Milnes Method – Adams-Bashforth Method.

Chapter 10: Sections 10.5 to 10.7.

Text Book:- Numerical Analysis

Numerical methods S. Arumugam and others, Scitech (2001)

unit - I :-

Interpolation :- Newton's Interpolation formula. central difference Interpolation - Lagrange's Interpolation formula - divided differences - Newton's divided differences formula - Inverse Interpolation - Hermit's Interpolating polynomial

Chapter 7 :- Section 7.1 to 7.7

unit - II :-

Numerical differentiation - derivates using Newton's forward, Backward central difference formula.

Chapter 8 :- Section 8.1 to 8.3

unit - III :-

Numerical Integration - Gaussian Quadrature formula - Numerical evaluation of double Intergrals

Chapter 8 :- 8.5 to 8.7

unit - IV :-

Numerical solutions of ordinary differential equations - Taylor's series Method -

picard's Method - Euler's Method . Runge kutta Method .

chapter 10 :- Sections 10.1 to 10.4

unit - V :-

predictor corrector Method - Milnes Method - Adams - Bashforth method .

~~chapter~~ Chapter 10 :- Section 10.5 to 10.7

8/07/15

Numerical Analysis

Difference Operator

Consider the function $f = f(x)$. Suppose we are given a table of values of the function at the points $x_0 , x_1 = x_0 + h , x_2 = x_0 + 2h , \ldots , x_n = x_0 + nh$.

let $f(x_0) = y_0 , f(x_1) = y_1 , \ldots , f(x_n) = y_n$

) forward Difference operator :-

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta y_0 = y_1 - y_0$$

) Backward difference operator :-

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla y_1 = y_1 - y_0$$

) Central difference operator :-

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

4) <u>Shift operator</u> :-

$$E \cdot f(x) = f(x+h)$$

5) <u>Averaging operator</u> :-

$$\mu f(x) = \frac{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}{2}$$

6) <u>Relations between operators</u>

* $\Delta = E - 1$ (or) $E = \Delta + 1$
* $\nabla = 1 - E^{-1}$
* $\delta = E^{1/2} - E^{-1/2}$
* $\mu = \dfrac{E^{1/2} + E^{-1/2}}{2}$
* $D = \dfrac{1}{h} \log E$

Pb from the forward difference table for the following data.

x	0	1	2	3	4
y	8	11	9	15	6

Soln:-

x	$y = f(x)$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
0	8	3			
1	11	$\cancel{+2}$ 3	-5		
2	9	-2	8	13	-36
3	15	6	-15	-23	
4	6	9			

Pb Find the missing Data.

x	0	1	2	3	4	5
y	2	6	12	20	30	a

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Soln:-

x	$y=f(x)$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
0	2					
		4				
1	6		2			
		6		0		
2	12		2		0	
		8		0		$a-42$
3	20		2		$a-42$	
		10		$a-42$		
4	30		$a-40$			
		$a-30$				
5	a					

$$\Delta^5 y_0 = 0$$

$$a-42 = 0$$

$$\boxed{a = 42}$$

Pb prove that $E\nabla = \nabla E = \Delta$

Soln:-

consider $E\nabla = E(1-E^{-1})$

$$= E - EE^{-1}$$

$$E\nabla = E - 1 \quad (\because EE^{-1} = 1)$$

|||ly,

$$\nabla E = (1-E^{-1})E$$

$$= E - E^{-1}E$$

$$\nabla E = E - 1 \quad (\because EE^{-1} = 1)$$

Hence $E\Delta = \nabla E = \Delta$

Pb prove that $\nabla\Delta = \delta^2$

proof:-

$$\delta^2 = (E^{1/2} - E^{-1/2})^2$$

$$= (E^{1/2} - E^{-1/2})(E^{1/2} - E^{-1/2})$$

$$= \left(E^{1/2} - \frac{1}{E^{1/2}}\right)\left(E^{1/2} - \frac{1}{E^{1/2}}\right)$$

$$= \frac{E'-1}{E}(E'-1) \qquad \left(\frac{E'-1}{E^{1/2}}\right)\left(\frac{E'-1}{E^{1/2}}\right)$$

$$= (1-E^{-1})(E-1)$$

$$= \Delta\nabla$$

$$E^{-1}E = 1$$

L.H.S $\Delta\nabla = (E-1)(1-E^{-1})$ $\qquad (1-E^{-1})(E-1)$

$$= E - EE^{-1} - 1 + E^{-1}$$

$$= E - 2 + E^{-1} \qquad a^2+b^2-2ab$$

$$= (E^{1/2} - E^{-1/2})^2 \qquad (a-b)^2$$

$$= \delta^2$$

L.H.S = R.H.S

Hence the proof.

Pb prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

proof:- $\Delta = E-1$, $\nabla = 1-E^{-1}$

R.H.S $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$

$$= \frac{(E-1)^2 - (1-E^{-1})^2}{(1-E^{-1})(E-1)}$$

$$= \frac{[(E-1)+(1-E^{-1})][(E-1)-(1-E^{-1})]}{E-1-1+E^{-1}}$$

$$= \frac{[(E-1)+(1-E^{-1})][E-2+E^{-1}]}{E-2+E^{-1}}$$

L.H.S = $\Delta + \nabla$

R.H.S = L.H.S

Hence the proof.

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Interpolation

Interpolation is the process of finding the most appropriate estimate for missing data. for making the most probile estimate be require the following assumption.

1) The frequency distribution is normal and not marked by sudden ups and down.

2). The Changes in the Series. are uniform with in a period.

Extrapolation:-

If we required information for future in which case the process of estimating the most appropriate value is known as "Extra polation".

Newton's Interpolation formulae (forward)

Let the function $y = f(x)$ take the values $y_0, y_1, \ldots y_n$ at the points $x_0, x_1, \ldots x_n$ where $x_i = x_0 + ih$. Then Newton's forward interpolation polynomial is given by,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0$$

where $x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h}$

Pb If $y(75) = 246$, $y(80) = 202$, $y(85) = 118$, $y(90) = 40$.

to find $y(79) = ?$

Soln:-

Newton's forward interpolation formula is,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0$$

where $x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h}$

$x_0 = 75$, $x = 79$, $h = 5$, $p = \frac{79-75}{5} = \frac{4}{5} = 0.8$

forward difference table.

x	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
75	246			
		-44		
80	202		-40	
		-84		46
85	118		6	
		-78		
90	40			

$$y_p = 246 + (0.8)(-44) + \frac{(0.8)(0.8-1)}{2\times 1}(-40) + \frac{(0.8)(0.8-1)(0.8-2)}{3\times 2\times 1} \times (46)$$

$$= 246 - 35.2 + \frac{(-0.16)}{2}(-40) + \frac{(-0.16)(-1.2)}{6} \times 46$$

$$= 246 - 35.2 + 3.2 + \frac{4.416}{3}$$

$$= 214 + \frac{4.416}{3}$$

$$= \frac{642 + 4.416}{3} \Rightarrow \frac{646.416}{3} \Rightarrow y_p = 215.472$$

$$\boxed{y_{0.8} = 215.472}$$

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Pb Find the cubic Polynomial which takes the following data.

x	0	1	2	3
$f(x)$	1	2	1	10

soln:-

$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, h = 1$

$$p = \frac{x - x_0}{h} = \frac{x-0}{1} = x$$

$$\boxed{p = x}$$

forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$$= 1 + x + \frac{x^2 - x}{\cancel{2}}(-\cancel{2}) + \frac{x(x^2 - 2x - x + 2)}{\cancel{6}}(\cancel{12}^{2})$$

$$= 1 + x - (x^2 - x) + 2x(x^2 - 3x + 2)$$

$$= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

$$y_x = 2x^3 - 7x^2 + 6x + 1$$

19/07/18

1) Find the following data and $f(9)$ using Newton's forward interpolation formula:

x	8	10	12	14	16
$f(x)$	1000	1900	3250	5400	8950

$y_p = 1405.85$

Soln: $x_0 = 8$, $x_1 = 10$, $x_2 = 12$, $x_3 = 14$, $x_4 = 16$

$$p = \frac{x - x_0}{h} = \frac{9-8}{2} = \frac{1}{2} = 0.5$$

$$\boxed{P = 0.5}$$

forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
8	1000				
		900			
10	1900		450		
		1350		350	
12	3250		800		250
		2150		600	
14	5400		1400		
		3550			
16	8950				

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 1000 + (0.5)(900) + \frac{(0.5)(0.5-1)}{2!}(450) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(350) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}$$

-0.5 -1.5 -2.5 $\times 25$

$$= 1000 + 450 - 56.25 + 21.875 - 9.77$$

$$\simeq 1405.855$$

$$y_{0.5} = 1405.86 \,//$$

2) find the value of y at $x = 21$ from the following data. $y_p : 0.35817$

x	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

Soln:-

$x_0 = 20, \quad x_1 = 23, \quad x_2 = 26, \quad x_3 = 29, \quad h = 3$

$$p = \frac{x - x_0}{h} = \frac{21 - 20}{3} = \frac{1}{3} = 0.33$$

$$\boxed{P = 0.3}$$

forward difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y$
20	0.3420			
		0.0487		
23	0.3907		-0.001	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

$$y_p = y_0 + p\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$= 0.3420 + (0.3)(0.0487) + \frac{(0.3)(0.3-1)}{2!}(-0.001)$$

$$+ \frac{(0.3)(0.3-1)(0.3-2)}{3!}(-0.0003)$$

$$= 0.3420 + 0.01461 + \frac{(0.3)(-0.7)(-0.001)}{2!} + \frac{(0.3)(-0.7)(-1.7)}{3!} \times (-0.000)$$

$$= 0.3420 + 0.01461 + 0.00021 - 0.00001785$$

$$= 0.3568$$

3) find f(2.5) using Newton's forward difference formula for the given

x	1	2	3	4	5	6
y	0	1	8	27	64	125

$y_p = 3.375$

soln:- $x_0=1,\ x_1=2,\ x_2=3,\ x_3=4,\ x_4=5,\ x_5=6,\ h=1$

$$p = \frac{x-x_0}{h} = \frac{2.5-1}{1} = \frac{1.5}{1} = 1.5$$

$$\boxed{p=1.5}$$

forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	0					
		1				
2	1		6			
		7		6		
3	8		12		0	
		19		6		0
4	27		18		0	
		37		6		
5	64		24			
		61				
6	125					

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{5!}\Delta^5 y_0$$

$$= 0 + (1.5)(1) + \frac{(1.5)(1.5-1)}{2!}(6) + \frac{(1.5)(1.5-1)(1.5-2)}{3!}6 + 0 + 0$$

$$= 0 + 1.5 + \frac{(1.5)(0.5)(6)}{2!} + \frac{(1.5)(0.5)(-0.5)}{3!}(6) + 0 + 0$$

$$= 1.5 + \cancel{4}\ 2.25 \cancel{+} - 0.375$$

$$= 3.375 \,//$$

4) population was recorded ...

village

year	1941	1951	1961	1971	1981	1991
population	2500	2800	3200	3700	4350	5225

Estimate the following population of the year 1945

Soln:- $x_0 = 1941, x_1 = 1951, x_2 = 1961, x_3 = 1971,$

$x_4 = 1981, x_5 = 1991, h = 10$

$$p = \frac{x - x_0}{h} = \frac{1945-1941}{10} = \frac{4}{10} = 0.4$$

~~Newto~~ difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	2500					
		300				
1951	2800		100			
		400		0		
1961	3200		100		50	
		500		50		-25
1971	3700		150		25	
		650		75		
1981	4350		225			
		875				
1991	5225					

forward interpolation formula

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0$$

$$= 2500 + (0.4)(300) + \frac{(0.4)(0.4-1)}{2!}(100) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(50) + \frac{(0.4)\ldots}{5!}$$

$= 2609.36$

backward interpolation formula $\quad p = \frac{x - x_n}{h} = \frac{1945-1991}{10} = \frac{-46}{10} = -4.6$

$\boxed{p = -4.6}$

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$= 5225 + (-4.6)(875) + \frac{(-4.6)(-4.6+1)}{2!}(225) + \frac{(-4.6)(-4.6+1)(-4.6+2)}{3!}(75)$$

$$+ \frac{(-4.6)(-4.6+1)(-4.6+2)(-4.6+3)}{4!}(25) + \frac{(-4.6)(-4.6+1)\ldots(-4.6+1)}{5!}(-25)$$

$=$

Newton's backward interpolation formulae

Let the function $y=f(x)$ take the values $y_0, y_1, \cdots y_n$ at the points $x_0, x_1, \cdots x_n$ where $x_i = x_0 + ih$. Then Newton's backward interpolation polynomial is given by,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$+ \frac{p(p+1)\cdots(p+(n-1))}{n!}\nabla^n y_n$$

where $x = x_n + ph$

$$\Rightarrow p = \frac{x - x_n}{h}$$

Pb 1) find the value of y from the following data at $\boxed{x = 2.65}$

x	-1	0	1	2	3
y	-21	6	15	12	3

Soln:- Since the value of $x (= 2.65)$ near the end of the data table we use Newton's interpolation formula. then $x = 2.65$, $x_n = 3$, $h = 1$, $p = \frac{x - x_n}{h}$

$$\Rightarrow \frac{2.65 - 3}{1} \Rightarrow \boxed{p = -0.35}$$

(-25)

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
-1	-21				
		27			
0	6		-18		
		9		6	
1	15		-12		0
		-3		6	
2	12		-6		
		-9			
3	3				

Newton's interpolation backward

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n +$$
$$\frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 3 + (-0.35)(-9) + \frac{(-0.35)(-0.35+1)}{2!}(-6) +$$
$$\frac{(-0.35)(-0.35+1)(-0.35+2)}{3!}(6) +$$
$$\frac{(-0.35)(-0.35+1)(-0.35+2)(-0.35+3)}{4!}(0)$$

$$= 3 + 3.15 + \frac{1.365}{2!} + \frac{2.25225}{3!} - \frac{35.810775}{4!}(0)$$

$$= 6.15 + 0.6825 - 0.375375 - 0$$

$y(2.65) = 6.45712$ //

b) The following data gives the point of an z The following data gives the temparature and x in C and lead Q is the temparature and x is the percentage of lead. using Newton's interpolation forward & backward

To find (i) Q when $x = 48$ forward

(ii) Q when $x = 84$ backward

internal

x	40	50	60	70	80	90
Q	184	204	226	250	276	304

Soln:- (i) Q when $x = 48$

$x = 48$, $x_0 = 40$, $h = 10$, $p = \frac{x - x_0}{h} = \frac{48-40}{10} = \frac{8}{10} = 0.8$

$\boxed{p = 0.8}$

x	θ	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

Newton's forward interpolation formula

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 +$$

$$\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0$$

$$= 184 + (0.8)20 + \frac{(0.8)(0.8-1)}{2!}(2)$$

$$= 184 + 16 + (0.8)(-0.2) = 184 + 16 - 0.16$$

$$= 199.84 \ //$$

$$p = \frac{x - x_n}{h} = \frac{84-90}{10}$$

Newton's backward interpolation formula $\boxed{p = -0.6}$

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n +$$

$$\frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$= 304 + (-0.6)(28) + \frac{(-0.6)(-0.6+1)}{2!}\,2 + 0 + 0 + 0$$

$$= 304 - 16.8 + (-0.6)(0.4)$$

$$= 304 - 16.8 + 0.24$$

$$y(84) = 286.96 \ //$$

Newton's forward Interpolation for [illegible] interval

Let $y = f(x)$ takes the values $y_0, y_1, \cdots y_n$ at the points $x_0, x_1, \ldots, x_n$. where $x_i = x_0 + ih$.

Then newton's forward interpolation polynomial is given by

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$+ \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0$$

where $x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h}$

proof:-

Let $\phi(x)$ be an interpolating polynomial of degree n which represents $y = f(x)$ in $x_0 \le x \le x_0 + nh$.

Then $\phi(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_0 - h)$
$+ a_3(x - x_0)(x - x_0 - h)(x - x_0 - 2h) + \cdots$
$\cdots + a_n(x - x_0)(x - x_0 - h)\cdots(x - x_0 - (n-1)h)$ — (1)

when $x = x_0$, $\phi(x_0) = f(x_0) = y_0$

from (1), $\phi(x_0) = a_0$

$\Rightarrow \boxed{y_0 = a_0}$

when $x = x_0 + h$, $\phi(x_0 + h) = f(x_0 + h) = f(x_1) = y_1$

from (1), $\phi(x_0 + h) = a_0 + a_1(x_0 + h - x_0)$

$y_1 = a_0 + a_1 h$

$y_1 = y_0 + a_1 h \quad [\because y_0 = a_0]$

$y_1 - y_0 = a_1 h \Rightarrow a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$

$\therefore a_1 = \frac{\Delta y_0}{h}$

when $x = x_0 + 2h$, $\phi(x_0+2h) = f(x_0+2h) = f(x_2) = y_2$

from ①,

$$\phi(x_0+2h) = a_0 + a_1(x_0+2h-x_0) + a_2(x_0+2h-x_0)(x_0+2h-x_0-h)$$

$$y_2 = a_0 + a_1 2h + a_2(2h)(h)$$

$$= a_0 + \frac{\Delta y_0}{h}(2h) + a_2 2h^2$$

$$2a_2h^2 = a_0 - 2\Delta y_0 + y_2 \quad [\because a_0 = y_0 ;\ f(x_2) = y_2]$$

$$2a_2h^2 = y_2 - y_0 - 2(y_1 - y_0)$$

$$= y_2 - y_0 - 2y_1 + 2y_0$$

$$= y_2 - 2y_1 + y_0$$

$$= (y_2 - y_1) - y_1 + y_0$$

$$= \Delta y_1 - (y_1 - y_0)$$

$$= \Delta y_1 - \Delta y_0$$

$$= \Delta(y_1 - y_0)$$

$$a_2 = \frac{\Delta^2 y_0}{2!\,h^2}$$

$|||^{ly}$ we get $a_3 = \frac{\Delta^3 y_0}{3!\,h^3}$

$$\vdots$$

$$a_n = \frac{\Delta^n y_0}{n!\,h^n}$$

$$\phi(x) = y_0 + (x-x_0)\frac{\Delta y_0}{h} + (x-x_0)(x-x_0-h)\frac{\Delta^2 y_0}{2h^2} + \dots + (x-x_0)(x-x_0-h)\cdots(x-x_0-(n-1)h)\frac{\Delta^n y_0}{n!\,h^n}$$

Since $\phi(x)$ is the interpolating polynomial which represents $y = f(x)$. Then $\phi(x)$ can be written as y,

y_p $y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots +$

$$\frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}\Delta^n y_n$$

where $x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h}$

Hence the proof.

20/07/18 **Newton's interpolation backward formula**

Let the function $y = f(x)$ take the values $y_0, y_1, \cdots y_n$ at the points $x_0, x_1, \cdots x_n$ where $x_i = x_0 + ih$.

Then newton's backward interpolation polynomial is given by,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots +$$

$$\frac{p(p+1)\cdots(p+(n-1))}{n!}\nabla^n y_n$$

where $x = x_n + ph$

$p = \frac{x - x_n}{h}$.

proof:-

Let $\phi(x)$ be an interpolating polynomial of degree n which represents $y = f(x)$ in $x_0 \le x \le x_0 + nh$.

Then,

$$\phi(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots +$$
$$a_n(x - x_n)(x - x_{n-1})\cdots(x - x_1) \quad \text{—— (1)}$$

when $x = x_n$ $\phi(x_n) = a_0 = f(x_n) = y_n$

from (1): $\boxed{a_0 = y_n}$

when $x = x_{n-1}$ $\phi(x_{n-1}) = f(x_{n-1}) = y_{n-1}$

from ①,

$$\phi(x_{n-1}) = a_0 + a(x_{n-1} - x_n) + a_2(0)$$

$$y_{n-1} = y_n + a_1(h) \quad (\because x_{n-1} - x_n = -h,\ a_0 = y_n)$$

$$a_1 = \frac{y_{n-1} - y_0}{-h}$$

$$= \frac{y_n - y_{n-1}}{h}$$

$$a_1 = \frac{\nabla y_n}{1!\,h}$$

when $x = x_{n-2}$, $\phi(x_{n-2}) = y_{n-2}$

from ① $\Rightarrow$ $\phi(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1}) + a_3(x_{n-2} - x_n)(x_{n-2} - x_{n-1})(x_{n-2} - x_{n-2})$

$$= a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$y_{n-2} = y_n + a_1(-2h) + a_2(-2h)(-h)$$

$$= y_n - 2a_1h - 2a_2h^2$$

$$2ah^2 = y_{n-2} - y_n + 2\frac{\nabla y_n}{h} \times h$$

$$= y_{n-2} - y_n + 2\nabla y_n$$

$$= y_{n-2} - y_n + 2(y_n - y_{n-1})$$

$$= y_{n-2} - y_n + 2y_{n-2}\, y_{n-1}$$

$$= y_{n-2} + y_n - 2y_{n-1}$$

$$= y_{n-2} + y_n - y_{n-1} - y_{n-1}$$

$$= y_{n-2} - y_{n-1} + y_n - y_{n-1}$$

$$= -[y_{n-1} - y_{n-2}] + y_n - y_{n-1}$$

$$= -\nabla y_{n-1} + \nabla y_n$$

$$= \nabla(y_n - y_{n-1})$$

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$$2a_2 h^2 = \nabla^2 y_n$$

$$a_2 = \frac{\nabla^2 y_n}{2!\,h^2}$$

$$a_3 = \frac{\nabla^3 y_n}{3!\,h^3}$$

$$\vdots$$

$$a_n = \frac{\Delta^n y_n}{n!\,h^n}$$

Substitude $a_1, a_2, a_3 \ldots a_n$ in (1)

$$\phi(x) = y_n + \frac{\nabla y_n}{1!\,h}(x-x_n) + \frac{\nabla^2 y_n}{2!\,h^2}(x-x_n)(x-x_{n-1})$$

$$+\cdots+ \frac{\nabla^n y_n}{n!\,h^n}(x-x_n)(x-x_{n-1})\cdots(x-x_1)$$

The gives newton's backward interpolation polynomial.

Since $\phi(x)$ is the interpolating polynomial with represent $y=f(x)$ then $\phi(x)$ can be written as, y

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$+ \frac{p(p+1)\cdots(p+(n-1))}{n!}\nabla^n y_n$$

where $p = \frac{x-x_n}{h}$

13/07/18 **Central difference interpolation formula**

Here, we introduce five Central difference interpolation formula Such as,

(i) Gauss forward interpolation formula

(ii) Gauss backward interpolation formula

(iii) Sterling's formula

(iv) Bessel's formula.

(v) Laplace Evertt's formula.

Consider the function $y=f(x)$ whose values of a collection of equally spaced points are given denote the middle point as x_0, show that set of equally spaced points are given below.

$x_0-3h,\ x_0-2h,\ x_0-h,\ x_0,\ x_0+h,\ x_0+2h,\ x_0+3h, \ldots$

Table of the points are represented as below.

x	..... x_0-3h	x_0-2h	x_0-h	x_0	x_0+h	x_0+2h	x_0+3h ...
$f(x)$	..... y_{-3}	y_{-2}	y_{-1}	y_0	y_{+1}	y_{+2}	y_{+3} ...

x	$f(x)=y$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
$x_3 = x_0-3h$	y_{-3}						
		$\delta y_{-5/2}$					
$x_2 = x_0-2h$	y_{-2}		$\delta^2 y_{-2}$				
		$\delta y_{-3/2}$		$\delta^3 y_{-3/2}$			
$x_1 = x_0-h$	y_{-1}		$\delta^2 y_{-1}$		$\delta^4 y_{-1}$		
		$\delta y_{-1/2}$		$\delta^3 y_{-1/2}$		$\delta^5 y_{-1/2}$	
x_0	y_0		$\delta^2 y_0$		$\delta^4 y_0$		$\delta^6 y_0$
		$\delta y_{1/2}$		$\delta^3 y_{1/2}$		$\delta^5 y_{1/2}$	
$x_1 = x_0+h$	y_{+1}		$\delta^2 y_1$		$\delta^4 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
$x_2 = x_0+2h$	y_{+2}		$\delta^2 y_2$				
		$\delta y_{5/2}$					
$x_3 = x_0+3h$	y_{+3}						

Difference table

x	$f(x)=y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
$x_3=x_0-3h$	y_{-3}						
		Δy_{-3}					
$x_2=x_0-2h$	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
$x_1=x_0-h$	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
$x_1=x_0+h$	y_{+1}		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
$x_2=x_0+2h$	y_{+2}		$\Delta^2 y_1$				
		Δy_2					
$x_3=x_0+3h$	y_{+3}						

The entries in the 1^{st} and 2^{nd} are same related to the operator relation is $\boxed{\delta = \Delta E^{-1/2}}$

$(\because f(x+h) = E)$

24/07/18 <u>Central difference operator</u> δ as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

If $f(x_i) = y_i$, then $y_1 - y_0 = \delta y_{1/2}$

$$y_2 - y_1 = \delta y_{3/2}$$

$$\vdots$$

$$y_n - y_{n-1} = \delta y_{n-1/2}$$

Pb: from the central difference table for the following data choosing $x = 35$ as orgin.

x	20	25	30	35	40	45
y	12	15	20	27	39	52

Soln:-

x	$p = \frac{x-35}{5}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
20	$\frac{20-35}{5} = -3$	12					
			3				
				2			
25	-2	15					
			5		0		
						3	
30	-1	20		2			
							-10
			7		3		
35	0	27		5		-7	
			12		-4		
40	1	39		1			
			13				
45	2	52					

$\Delta y_{-3} = 3$ | $\Delta y_{-1} = 7$ | $\Delta y_1 = 13$

$\Delta y_{-2} = 5$ | $\Delta y_0 = 12$

Δy_{-3}	$\Delta^2 y_{-3} = 2$	$\Delta^3 y_{-3} = 0$	$\Delta^4 y_{-3} = 3$	$\Delta^5 y_{-3} = -10$
	$\Delta^2 y_{-2} = 2$	$\Delta^3 y_{-2} = 3$	$\Delta^4 y_{-2} = -7$	
	$\Delta^2 y_{-1} = 5$	$\Delta^3 y_{-1} = 4$		
	$\Delta^2 y_0 = 1$			

Gauss forward interpolation formula

W.K.T Newton's forward interpolation formula is,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0 \quad \text{--- (1)}$$

where $x = x_0 + ph$

we have, $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$

$$\boxed{\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$|||^{ly}$ $\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$

$\vdots$

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$|||^{ly}$, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc

Substitude these values in eqn (1).

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0 \quad \text{--- (1)}$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} - \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \cdots$$

$$+ \frac{p(p-1)\cdots(p-(n-1))}{n!}(\Delta^n y_{-1} + \Delta^{n+1} y_{-1}) -$$

$$= y_0 + p\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \left(\binom{p}{2}+\binom{p}{3}\right)\Delta^3 y_{-1} + \left(\binom{p}{3}+\binom{p}{4}\right)\Delta^4 y_{-1} + \left(\binom{p}{4}+\binom{p}{5}\right)\Delta^5 y_{-1} + \cdots$$

$$= y_0 + p\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \binom{p+1}{4}\Delta^4 y_{-1} + \binom{p+1}{5}\Delta^5 y_{-1} + \cdots$$

$$= y_0 + p\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \binom{p+1}{4}\left[\Delta^4 y_{-2} + \Delta^5 y_{-2}\right] + \binom{p+1}{5}\left[\Delta^5 y_{-2} + \Delta^6 y_{-2}\right] + \cdots$$

$$= y_0 + \binom{p}{1}\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \cdots$$

$$y_p = y_0 + \binom{p}{1}\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \binom{p+1}{4}\Delta^4 y_{-2} + \binom{p+2}{5}\Delta^5 y_{-2} + \cdots$$

This formula is known as Gauss forward interpolation formula.

Pb Apply Gauss forward interpolation formula to obtain $f(x)$ at $x = 3.5$ from the table below.

x	2	3	4	5
$f(x)$	2.626	3.454	4.784	6.986

x_0 means mid point

Soln:-

$$y_p = y_0 + \binom{p}{1}\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \binom{p+1}{4}\Delta^4 y_{-2} + \cdots$$

Soln:- $x = 3.5$, $x_0 = 3$, $h = 1$, $p = \frac{0.5}{1} = 0.5$

x	$p = \frac{x-3}{1}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	-1	2.626			
			0.828		
3	0	3.454		0.502	
			1.33		0.37
4	+1	4.784		0.872	
			2.202		
5	+2	6.986			

Gauss forward interpolation formula:

$$\boxed{y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}}$$

$$y_{0.5} = 3.454 + (0.5)(1.33) + \frac{(0.5)(-0.5)}{2!}(0.502) + \frac{(1.5)(0.5)(-0.5)}{3!}(0.37)$$

midpoint: so 3ku corresponding value eduka

$$= 3.454 + 0.665 - 0.06275 - 0.023125$$

$$y_{(0.5)} = 4.03312.$$

26/07/18

Q: using Gauss forward interpolation formula find $f(30)$ from the following table.

internal

x	21	25	29	33	37
$f(x)$	18.4708	17.8144	17.1070	16.3432	15.5154

Soln:- $x = 30$, $x_0 = 29$, $h = 4$, $p = \frac{x - x_0}{h} = \frac{30-29}{4} = \frac{1}{4}$

$$\boxed{p = 0.25}$$

$$f(30) \doteq y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)(p)(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)(p)(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$

x	$p=\frac{x-29}{4}$	y	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
21	-2	18.4708				
			-0.6564			
25	1	17.8144		-0.051		
			-0.7074		-0.0054	
29	0	17.1070		-0.0564		-0.0022
			-0.7638		-0.0076	
33	1	16.3432		-0.064		
			-0.8278			
37	2	15.5154				

$$= 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(-0.75)}{2!}(-0.0564)$$

$$+ \frac{(0.25)(-0.75)(1.25)}{3!}(-0.0076) +$$

$$\frac{(1.25)(0.25)(-0.75)}{4!}(-0.0022)$$

$$= 17.1070 - 0.19095 + 0.0052875 + 0.000296875 - 0.0000375976$$

$$= 16.92159$$

$f(30) \simeq 16.9216$

Pb Given $f(2)=10$, $f(1)=8$, $f(0)=5$, $f(-1)=10$ find $f(1/2)$ by Gauss forward formula.

Soln:- $x = 1/2 \quad x_0 = 1 \quad h = -1, \quad p = \frac{x-x_0}{h} = \frac{1/2 - 1}{-1} = \frac{0.5-1}{-1} = 0.5$

x	$p=\frac{x-1}{-1}$	f	Δy	$\Delta^2 f$	$\Delta^3 f$
2	-1	10			
			-2		
1	0	8		-1	
			-3		9
0	1	5		8	
			5		
-1	2	10			

$$f(\tfrac{1}{2}) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)(p)(p-1)}{3!}\Delta^3 y_{-1}$$

$$= (8) + (0.5)(-3) + \frac{(0.5)(0.5-1)(-1)}{2!} + \frac{(0.5+1)(0.5)(0.5-1)}{3!}\cdot(9)$$

$$= 8-1.5 + \frac{(0.5)(-0.5)(-1)}{2!} + \frac{(1.5)(0.5)(-0.5)}{3!}(9)$$

$$= 8-1.5+\frac{0.25}{2!}+\frac{(-0.375)}{\cancel{6}\,2}\overset{3}{\cancel{(9)}}$$

$$= 8-1.5+0.125 - \frac{1.125}{2}$$

$$= 8-1.5+0.125-0.5625$$

$$= 6.0625$$

$f(0.5) = 6.0625$ //

Gauss backward interpolation formula

W.K.T Newton's forward interpolation formula is,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)\dots(p-(n-1))}{n!}\Delta^n y_0 \quad \text{—(1)}$$

where $x = x_0 + ph$

we have $\Delta y_0 - \Delta y_1 = \Delta^2 y_{-1}$

$$\Delta y_0 = \Delta y_1 + \Delta y_{-1}$$

$$|||^{ly} \quad \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

substitude these values in eqn ①

$$① \Rightarrow y_p + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2}) + \dots$$

$$= y_0 + \binom{p}{1}(\Delta y_{-1} + \Delta^2 y_{-1}) + \binom{p}{2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \binom{p}{3}(\Delta^3 y_{-2} + \Delta^4 y_{-2}) + \dots$$

$$= y_0 + \binom{p}{1}\Delta y_{-1} + \left(\binom{p}{1} + \binom{p}{2}\right)\Delta^2 y_{-1} + \left(\binom{p}{2} + \binom{p}{3}\right)\Delta^3 y_{-2} + \left(\binom{p}{3} + \binom{p}{4}\right)\Delta^4 y_{-2} + \dots$$

$$y_p = y_0 + \binom{p}{1}\Delta y_{-1} + \binom{p+1}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-2} + \binom{p+2}{4}\Delta^4 y_{-2} + \dots$$

7/18 pb Using Gauss backward interpolation formula find Sin 45° from the following data. ✓ internal

x°	20	30	40	50	60	70
$\sin x^\circ = y$	0.342	0.502	0.642	0.766	0.866	0.939

Soln:- $x = 45^\circ$, $x_0 = 50$, $h = 10$, $p = \frac{45-50}{10} = -0.5$

$$y_p = y_0 + p\nabla y_{-1} + \frac{p(p-1)}{2!}\nabla^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\nabla^3 y_{-2} + \dots$$

$$= 0.766 + (-0.5)(0.124) + \frac{(0.5)(-0.5)}{2!}(-0.024)$$

$$+ \frac{(0.5)(-0.5)(-1.5)}{3!}(-0.008) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!}(0.00\ldots)$$

$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)(-2.5)}{5!}(0.017)$$

x	$p=\frac{x-50}{10}$	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
20	-3	0.342					
			0.16				
30	-2	0.502		-0.02			
			0.14		0.004		
40	-1	0.642		-0.016		-0.012	
			0.124		-0.008		0.017
50	0	0.766		-0.024		0.005	
			0.1		-0.003		
60	1	0.866		-0.027			
			0.073				
70	2	0.939					

$$= 0.766 - 0.062 + 0.003 - 0.005 - 0.0001953125 - 0.00019921875$$

$$= 0.706105$$

P3 Appy Gauss backward interpolation formula to find $y(25)$ for the following table.

x	20	24	28	32
y	2854	3162	3544	3992

Ans:- 3250.875

$x_0 = 24$ (or) $\frac{24+28}{2}$

Soln:- $x = 25$; $x_0 = 24$, $h = 4$

$$p = \frac{25-24}{4} = \frac{1}{4} = 0.25$$

$$\boxed{p = 0.25}$$

x	$p=\frac{x-50}{10}$	y	∇y	$\nabla^2 y$	$\nabla^3 y$
20	-3	2854			
			308		
24	-2.6	3162		74	
			382		-8
28	-2.2	3544		66	
			448		
32	-1.8	3992			

$$y_p = y_0 + p\nabla y_{-1} + \frac{p(p-1)}{2!}\nabla^2 y_{-1} + \frac{(p+1)(p(p-1)}{3!}\nabla^3 y_{-2}$$

$= \cancel{2854}\ 3162 + (0.25)(\ldots) \frac{\ldots}{2!}$

$+ \frac{(0.25+1)(0.25)(0.25-1)}{3!}(-8)$

$= 3162 + 77 + \frac{(0.25)(-0.75)}{2!}(66) + \frac{(1.25)(0.25)\ldots}{6}$

$= 3232.8125 - 0.3125$

$= 3232.5$

Sterling's interpolation formula

Gauss forward interpolation formula is,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \quad (1)$$

Gauss backward interpolation formula is,

$$y_p = y_0 + p\nabla y_{-1} + \frac{p(p+1)}{2!}\nabla^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\nabla^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\nabla^4 y_{-2} + \cdots \quad (2)$$

Taking the mean of (1) & (2)

$$y_p = y_0 + \frac{p}{2}\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{1}{2}\left[\frac{p(p-1)+(p+1)p}{2}\right]\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \left[\frac{(p+1)p(p-1)(p-2)}{4!} + \frac{(p+2)(p+1)p(p-1)}{4!}\right]\frac{\Delta^4 y_{-2}}{2} + \cdots$$

$$y_p = y_0 + \frac{p}{2}\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{1}{2}\cdot\frac{p(p^2-1)}{3!}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right] + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

This is called Stirilling formula

Note:-

1) If interpolation is desired near the begining of the table we use Newton's forward interpolation formula. Since higher order central difference do not exists at the begining of the table.

2) If the interpolation is desired near the ending of the table we use Newton's backward interpolation formula.

3) Gauss farward interpolation formula is best result for $0 < p < 1$.

4) Gauss backward interpolation formula is best result for $-1 < p < 0$.

5) To find an interpolated value near the middle of the table Stirilling formula gives most accurate result for $\frac{-1}{4} \le p \le \frac{1}{4}$.

6) Bessel's formula and everett's formula give the most accurate result for $\frac{1}{4} \le p \le \frac{3}{4}$.

Pb Apply stirilling formula to find $y(25)$ for the following data.

x	20	24	28	32
y	2854	3162	3544	3992

Soln:- $x = 25$, $x_0 = 24$, $h = 4$.

$$p = \frac{25-24}{4} = \frac{1}{4} = 0.25$$

$$\boxed{p = 0.25}$$

x	$p = \frac{x-24}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	-1	2854			
			308		
24	0	3162		74	
			382		-8
28	1	3544		66	
			448		
32	2	3992			

$$y_p = y_0 + \frac{p}{2}\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p^2-1)}{3!}\left[\Delta^3 y_{-1} + \Delta^3\right.$$

$$= 3162 + \frac{0.25}{2}[382+308] + \frac{(0.25)^2}{2}(\cancel{64}\,37) + \frac{1}{2}\frac{(0.25)(0.}{3!}$$

$$= 3162 + 86.25 + 2.3125 + 0.15625$$

$$= 3250.71875$$

Pb: using Stirilling formula compute y_{35} given that $y_{10} = 600$, $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 348$, $y_{50} = 243$

Ans: 373

30/07/18 **Bessel's formula**

Gauss farward interpolation formula is,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \cdots \quad (1)$$

W.k.T $\Delta y_0 = y_1 - y_0$

$y_0 = y_1 - \Delta y_0$

$y_{-1} = y_0 - \Delta y_{-1}$

$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$

|||ly

$\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$

eqn (1) can be written as,

$$y_p = \left(\frac{y_0}{2} + \frac{y_0}{2}\right) + p\Delta y_0 + \left[\frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\cdot\frac{p(p-1)}{2!}\Delta^2 y_{-1}\right] + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$= \frac{y_0}{2} + \frac{y_0}{2} + p(\Delta y_0) + \frac{1}{2}\left(\frac{p(p-1)}{2}\Delta^2 y_{-1}\right) + \frac{1}{2}\frac{p(p-1)}{2}[\Delta^2 y_0 - \Delta^3 y_{-1}] + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$= \frac{y_0}{2} + \frac{1}{2}(y_1 - \Delta y_0) + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}[\Delta^2 y_0 - \Delta^3 y_{-1}] + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$= \left(\frac{y_0+y_1}{2}\right) + (p-\tfrac{1}{2})\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}[\Delta^2 y_{-1} + \Delta^2 y_0] +$$

$$\left(\frac{p(p-1)}{2!}\right)\left(\frac{-1}{2} + \frac{p+1}{3}\right)\Delta^3 y_{-1} + \cdots$$

$$y_p = \left(\frac{y_0+y_1}{2}\right) + (p-\tfrac{1}{2})\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}[\Delta^2 y_{-1} + \Delta^2 y_0] +$$

$$\left(\frac{p(p-1)}{2!}\right)\left(\frac{-1}{2} + \frac{p+1}{3}\right)\Delta^3 y_{-1} + \cdots$$

$$y_p = \left(\frac{y_0+y_1}{2}\right) + (p-\tfrac{1}{2})\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}[\Delta^2 y_{-1} + \Delta^2 y_0]$$

$$+ \frac{p(p-1)(p-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

This called Bessel's formula.

Pb Apply Bessel's formula to find $y(25)$ for the following table:

x	20	24	28	32
y	2854	3162	3544	3992

Soln: $x = 25$, $x_0 = 24$ $\quad p = \frac{x - x_0}{h} = \frac{25-24}{4} = \frac{1}{4} = 0.2$

x	$p = \frac{x-24}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	-1	2854			
			308		
24	0	3162		74	
			382		-8
28	1	3544		66	
			448		
32	2	3992			

$$y_p = \left(\frac{y_0+y_1}{2}\right) + \left(p-\frac{1}{2}\right)\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^2 y_0\right] + \frac{p(p-1)}{3!}\left(\frac{-1}{2} + \frac{p+1}{3}\right)\Delta^3 y_{-1}$$

$$= \left(\frac{3544+3162}{2}\right) + \left(0.25-\frac{1}{2}\right)(382) + \frac{1}{2}\frac{(0.25)(0.25-1)}{2!}\left[66+74\right] + \frac{(0.25)(0.25-1)}{3!}\left[\frac{-1}{2} + \frac{(0.25+1)}{3}\right](-8)$$

$$= 3353 + (-0.25)(382) + \frac{1}{2}\left[\left(\frac{-0.1875}{2}\right)(140) + \left(\frac{-0.1875}{3!}\right)(-0.5+0.417)(-8)\right]$$

$$= 3353 - 95.5 - 6.5625 + 0.03125(-0.083)(-8)$$

$$= 3250.95825$$

Pb Apply Bessel's formula to find $y(1.95)$ for the following table

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(x)$	2.979	3.144	3.283	3.391	3.463	3.997	4.491

which are interpolation formula can be used here which is more appropriate? Give reasons.

interne

soln:- $x = 1.95$, $x_0 = 1.9$, $h = 0.1$, $p = \frac{0.05}{0.1} = 0.5$

x	$p = \frac{x-1.9}{0.1}$	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
1.7	-2	2.979						
			0.165					
1.8	-1	3.144		-0.026				
			0.139		-0.005			
1.9	0	3.283		-0.031		0		
			0.108		-0.005		0.503	
2.0	1	3.391		-0.036		0.503		-2.006
			0.072		0.498		-1.503	
2.1	2	3.463		0.462				
			0.534		-0.502			
2.2	3	3.997		-0.04				
			0.494					
2.3	4	4.491						

$$y_p = \left(\frac{y_0+y_1}{2}\right) + \left(p-\frac{1}{2}\right)\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^2 y_0\right] + \frac{p(p-1)}{2!}\left(\frac{-1}{2} + \frac{p+1}{3}\right)\Delta^3 y_{-1} +$$

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$$\frac{(p+1)(p)(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\right)$$

$$=\left(\frac{3.283+3.391}{2}\right)+(0.5-0.5)(0.108)+\frac{1}{2}\frac{(0.5)(-0.5)}{2!}(-0.031-0.036)$$

$$+\frac{(0.5)(-0.15)}{2!}\left(-0.5+\frac{1.5}{3}\right)(-0.005)+$$

$$\frac{(1.5)(0.5)(-0.5)}{4!}(-1.5)\left[\frac{0+0.503}{2}\right]$$

$$=3.337+0.004187570+0.00589453$$

$$=3.3470082$$

31/07/18 **Laplace Everett's formula :-**

Gauss forward interpolation formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\binom{p+1}{4}\Delta^4 y_{-2} + \binom{p+2}{5}\Delta^5 y_{-2} + \dots \quad (1)$$

Also, $\Delta y_0 = y_1 - y_0$

$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$

$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$$

Substitude these values odd differences in (1)

$$y_p = y_0 + \binom{p}{1}(y_1 - y_0) + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}(\Delta^2 y_0 - \Delta^2 y_{-1}) +$$

$$\binom{p+1}{4}\Delta^4 y_{-2} + \binom{p+2}{5}(\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \dots$$

$$=(1-p)y_0 + p y_1 + \left[\binom{p}{2} - \binom{p+1}{3}\right]\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^2 y_0 +$$

$$\left[\binom{p+1}{4} - \binom{p+2}{5}\right]\Delta^4 y_{-2} + \binom{p+2}{5}\Delta^4 y_{-1} + \dots$$

$$y_p = (1-p)y_0 + py_1 - \left(\frac{p(p-1)(p-2)}{3!}\right)\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^2 y_0 -$$

$$\left(\frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\right)\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^4 y_1 \dots$$

Change the tearms with negative signs, put $p = 1-q$,

we get,

$$y_p = qy_0 + py_1 + \frac{q(q^2-1)}{3!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^2 y_0 + \frac{q(q^2-1)}{5!}\Delta^4 y_{-2}$$

$$+ \frac{p(p^2-1^2)}{5!}\Delta^4 y_{-1} + \dots$$

$$+ py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \dots$$

∴ This is known as Laplace Evertt's formula.

Pb Using Laplace Evertt's formula to find $y(25)$ for the following table.

x	20	24	28	32
y	2854	3162	3544	3992

Soln: $x = 25$; $x_0 = 24$; $p = 1/4 = 0.25$, $q = 1-p = 0.75$

x	$p = \frac{x-24}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	-1	2854			
			308		
24	0	3162 y_0		74	
			382		-8
28	1	3544		66	
			448		
32	2	3992			

$$y_p = qy_0 + py_1 + \frac{q(q^2-1)}{3!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^2 y_0 + \quad (1)$$

$$\frac{q(q^2-1)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-}$$

$$= (0.75) + (0.25)(2854) + \frac{(0.75)((0.75)^2-1)}{3!}(66)$$

01/08/18 **Legrange's Interpolation formula**

Let $y_0, y_1, y_2 \cdots y_n$ be the values of $f(x)$ at $x_0, x_1, x_2 \cdots x_n$ (not necessarly at equal interval) Then an interpolating polynomial $\phi(x)$ for $f(x)$ is given by,

$$\phi(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}(y_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1$$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}\cdot y_n$$

proof:-

Since n values of $f(x)$ are given we can assume $f(x)$ to be a polynomial of degree $(n-1)$

Let $\phi(x) = A_0(x-x_1)(x-x_2)\cdots(x-x_n) + A_1(x-x_0)(x-x_2)\cdots(x-x_n)$

$$+\cdots+A_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad (1)$$

when $x = x_0$, $y_0 = A_0(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)$

$$A_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

III^{ly}, when $x = x_1, x_2, \cdots x_n$

$$A_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}$$

$$A_2 = \frac{y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_n)}$$

$\vdots$

$$A_n = \frac{y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Substitude in ① we get,

$$\phi(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}(y_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}(y_1)$$

$$+\cdots+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

Since $\phi(x)$ is the interpolating polynomial which represent $f(x)$.

Hence Legranges formula becomes,

$$y = f(x) = \frac{(x-x_1)\cdots(x-x_n)}{(x_0-x_1)\cdots(x_0-x_n)}(y_0) + \frac{(x-x_0)\cdots(x-x_n)}{(x_1-x_0)\cdots(x_1-x_n)}(y_1)$$

$$+\cdots+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}(y_n)$$

03/08/18

Pb use lagrange's formula to find the value of y at $x=6$ from the following data

x	3	7	9	10
y	168	120	72	63

Soln: Hence Lagrange's formula for four set data in,

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1)$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

when $x = 6$

$$y(6) = \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}(168) + \frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)} \times (120)$$

$$+ \frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}(72) + \frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}(63)$$

$$= \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)} \times 168 + \frac{(3)(-3)(-4)}{(4)(-2)(-3)}(120)$$

$$+ \frac{(3)(-1)(-4)}{(6)(2)(-1)}(72) + \frac{(3)(-1)(-3)}{(7)(3)(1)}(63)$$

$$= 12 + 180 - 72 + 27$$

$$= 147 \text{ //}$$

Pb Use lagrange's formula to fit a polynomial to the following data.

x	0	1	3	4
y	-12	0	6	12

find the value of y when $x = 2$

Soln:-

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

$$= \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) + \frac{(x)(x-3)(x-4)}{(-1)(1-3)(1-4)}(0)$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3)(3-1)(3-4)}(6) + \frac{(x)(x-1)(x-3)}{(4)(4-1)(4-3)}(12)$$

$$= (x-1)(x-3)(x-4) - x(x-1)(x-4) + x(x-1)(x-3)$$

$$= (x-1)(x-3)(x-4) - x(x^2-4x-x+4) + x(x^2-3x-x+3)$$

$$= (x^2-3x-x+3)(x-4) - (x^3-4x^2-x^2+4x) + (x^3-3x^2-x^2+3x)$$

$$= (x^3-3x^2-x^2+3x-4x^2-12x-4x-12) - (x^3-4x^2-x^2+4x) + (x^3-4x^2+3x)$$

$$= (x^3-8x^2-13x-12) - (x^3-5x^2+4x) + (x^3-4x^2+3x)$$

$$= \cancel{x^3} - 8x^2 - 13x - 12 - \cancel{x^3} + 5x^2 - 4x + x^3 - 4x^2 + 3x$$

$$= x^3 - 7x^2 - 18x - 12$$

$x = 2 \Rightarrow (2)^3 - 7(2)^2 - 18(2) - 12 = 8 - 7(4) - 36 - 12$

$$\boxed{y(2) = 4}$$

Pb use Lagrange's interpolation formula to find the value of y when $x = 10$ if the following value of x & y are given

x	5	6	9	11
y	12	13	14	16

Soln:-

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}($$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x}($$

when $x = 10$, $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$

$y_0 = 12$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$

$$= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

$$\boxed{y = 14.6667}$$

Pb use lagrange's interpolation formula to find $f(x)$ when $x=0$ given the following data,

x	-1	-2	2	4
$f(x)$	-1	-9	11	69

Soln:- $x=0$, $x_0=-1$, $x_1=-2$, $x_2=2$, $x_3=4$

$y_0=-1$, $y_1=-9$, $y_2=11$, $y_3=69$

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

$$= \frac{(0+2)(0-2)(0-4)}{(-1+2)(-1-2)(-1-4)}(-1) + \frac{(0+1)(0-2)(0-4)}{(-2+1)(-2-2)(-2-4)}(-9)$$

$$+ \frac{(0+1)(0+2)(0-4)}{(2+1)(2+2)(2-4)}(11) + \frac{(0+1)(0+2)(0-2)}{(4+1)(4+2)(4-2)}(69)$$

$$= \frac{(2)(-2)(-4)}{(1)(-3)(-5)}(-1) + \frac{(1)(-2)(-4)}{(-1)(-4)(-6)}(-9) + \frac{(1)(2)(-4)}{(3)(4)(-2)}(11) + \frac{(1)(2)(-2)}{(5)(6)(2)}(69)$$

$$y = -1.06667 + 3 + 3.6667 - 4.6$$

$$\boxed{y = 1.}$$

Divided Differences

Definition:-

Let $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ be a given set of $(n-1)$ points. The first divided differences are defined by the following relations

$$[x_0 - x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_1 - x_2] = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\vdots$$

$$[x_{n-1}, x_n] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

The second, divided differences are defined by,

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} \text{ and so on}$$

The third divided differences are defined by,

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on}$$

The divided differences are denoted by $\Delta, \Delta^2, \Delta^3, \ldots$

The divided differences table is given below,

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		$[x_0 x_1]$			
x_1	y_1		$[x_0 x_1 x_2]$		
		$[x_1 x_2]$		$[x_0 x_1 x_2 x_3]$	
x_2	y_2		$[x_1 x_2 x_3]$		$[x_0, x_1, x_2, x_3, x_4]$
		$[x_2 x_3]$		$[x_1, x_2, x_3, x_4]$	
x_3	y_3		$[x_2 x_3 x_4]$		
		$[x_3 x_4]$			
x_4	y_4				

The divided differences are independent of the order of arrangements.

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = [x_1, x_0]$$

IIIly

$$[x_0, x_1, x_2] = [x_1, x_2, x_0] = [x_2, x_0, x_1]$$ //

06/08/18 Pb. Using Newton's divided difference formula evaluate $f(8)$ given that

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Soln:-

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
4 (x_0)	48 (y_0)					
		$\frac{y_1-y_0}{x_1-x_0}$ $\frac{52}{1}=52$				
5 (x_1)	100 (y_1)		$\frac{y_1-y_0}{x_2-x_0}$ 15			
		97 $\frac{y_2-y_1}{x_2-x_1}$		1 $\frac{y_1-y_0}{x_3-x_0}$		
7 (x_2)	294 (y_2)		21 $\frac{y_2-y_1}{x_3-x_1}$		0	
		202 $\frac{y_3-y_2}{x_3-x_2}$				0
10 (x_3)	900 (y_3)		27	1 $\frac{y_2-y_1}{x_4-x_1}$		
		310 $\frac{y_4-y_3}{x_4-x_3}$	$\frac{y_3-y_2}{x_4-x_2}$		0	
11 (x_4)	1210 (y_4)		33	1		
		409	$\frac{y_4-y_3}{x_5-x_3}$	$\frac{y_3-y_2}{x_5-x_2}$		
13 (x_5)	2028 (y_5)					

Newton's divided difference formula is

$$f(x) = f(x_0) + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})[x_0, x_1, x_2, \dots x_n]$$

$$= 48 + (8-4)[4,5] + (8-4)(8-5)[4,5,7] + (8-4)(8-5)(8-7)[4,5,7,10] + (8-4)(8-5)(8-7)(8-10)(8-11)[4,5,7,10,11,13]$$

$$= 48 + 4(52) + (4)(3)(15) + 4(3)(1)(1) + 4(3)(1)(-2)(0) + 0$$

$$= 48 + 208 + 1080 + 2$$

$$= 448$$ //

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Find the equation of the cubic curve which passess through the points (4, −43), (7, 83) (9, 327) (12, 1053) to find newton's divided difference formula. Hence find f(10)

Soln:

x	y	Δ	Δ^2	Δ^3
4 (x_0)	−43 (y_0)			
		42		
7 (x_1)	83 (y_1)		16	
		122		1
9 (x_2)	327 (y_2)		24	
		242		
12 (x_3)	1053 (y_3)			

Newton's divided difference formula is,

$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] +$

$(x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$

$= x^3 - 4x^2 - 7x - 15$

$= (-43) + (10 - (4))[4, 7] + (10 - 4)(10 - 7)(4, 7, 9) +$

$(10 - 4)(10 - 7)(10 - 9)(4, 7, 9, 12)$

$= -43 + (6)(42) + (6)(3)(16) + (6)(3)(1)(1)$

$= -43 + 252 + 288 + 18$

$= 515$

Pb Find the divided difference table for the following data.

x	−1	0	2	4	5
y	0	1	9	65	126

Soln:

x	y	Δ	Δ^2	Δ^3	Δ^4
−1	0				
		1			
0	1		1		
		4		1	
2	9		6		0
		28		1	
4	65		11		
		61			
5	126				

09/08/18 **Inverse interpolation**

The process of estimating the value of x for some value of y which is not in the table is called inverse interpolation.

There are two types are,

1) Lagrange's method
2) Interative method.

Lagranges method

Interchanging the variables x and y is lagranges formula we get,

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}(x_0) + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}(x_1) + \frac{(y-y_0)(y-y_1)\cdots(y-y_n)}{(y_2-y_0)(y_2-y_1)\cdots(y_2-y_n)}(x_2) + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}(x_n)$$

Pb find the value of x correct to one decimal place for which $y=7$ given

x	1	3	4
y	4	12	19

Soln:-

$x_0=1,\ x_1=3,\ x_2=4,\ y_0=4,\ y_1=12,\ y_2=19$

$$x = \frac{(y-y_1)(y-y_2)\ \cancel{(y-y_3)}}{(y_0-y_1)(y_0-y_2)}(x_0) + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}(x_1) + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}(x_2)$$

$$= \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3) + \frac{(7-4)(7-12)}{(19-4)(19-12)}(4)$$

$$= \frac{(-5)(-12)}{(-8)(-15)}(1) + \frac{(3)(-12)}{(8)(-7)}(3) + \frac{(3)(-5)}{(15)(7)}(4)$$

$$= \frac{60}{120}(1) + \frac{36}{56}(3) + \frac{(-15)}{105}(4)$$

$$= 0.5 + 1.928571429 - 0.571428571$$

$$= 1.8571$$

Pb The value of x and u_x are given below.

x	5	6	9	11
u_x	12	13	11	16

find the value of x when $u_x = 15$.

Soln:- $x_0 = 5,\ x_1 = 6,\ x_2 = 9,\ x_3 = 11,\ ux_0 = 12,\ ux_1 = 13,\ ux_2 = 11$

$ux_3 = 16$

$$u_{x(15)} = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}(x_0) + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}(x_1)$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}(x_2) + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}(x_3)$$

$$= \frac{(15-13)(15-11)(15-16)}{(12-13)(12-11)(12-16)}(5) + \frac{(15-12)(15-11)(15-16)}{(13-12)(13-11)(13-16)}(6) +$$

$$\frac{(15-12)(15-13)(15-16)}{(11-12)(11-13)(11-16)}(9) + \frac{(15-12)(15-13)(15-11)}{(16-12)(16-13)(16-11)}(11)$$

$$= \frac{-8}{4}(5) + \frac{(-12)}{-6}(6) + \frac{-6}{-10}(9) + \frac{24}{60}(11)$$

$$= -10 + 12 + 5.4 + 4.4$$

$$= 11.8$$

Iterative method

Newtons forward difference formula is:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\cdots(p-n-1)}{n!}\Delta^n y_0$$

$$p = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p(p-1)}{2!}\Delta^2 y_0 - \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 - \cdots\right]$$

Neglecting the second and higher order differences we obtain the first approximation to p given by

$$p_1 = \frac{1}{\Delta y_0}(y_p - y_0)$$

To find the second approximation to p we retain term with second difference and replace p by p_1.

$$\therefore p_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 y_0\right]$$

find the final third approximation we retain the terms upto third order difference and replace p by

$$\therefore p_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 y_0 - \frac{p_2(p_2-1)}{3!}\Delta^3 y_0\right]$$

continue this process till the successive values of p are approximately equal.

Pb Tabulate $y = x^3$ for $x = 2, 3, 4, 5$ and calculate the cube root of 10 correct to 3 decimal places

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	8			
		19		
3	27		18	
		37		6
4	64		24	
		61		
5	125			

$$p_1 = \frac{1}{\Delta y_0}(y_p - y_0) = \frac{1}{19}(10-8) = \frac{1}{19}(2)$$

$$\boxed{p_1 = 0.1053}$$

$$p_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 y_0\right]$$

$$= \frac{1}{19}\left[10-8 - \frac{(0.1053)(0.1053-1)}{2!}(18)\right]$$

$$= \frac{1}{19}\left[2 + 0.84791\right] = \frac{1}{19}\left[2.84791\right]$$

$$\boxed{p_2 = 0.14989}$$

$$P_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{P_2(P_2-1)}{2!}\Delta^2 y_0 - \frac{P_2(P_2-1)(P_2-2)}{3!}\Delta^3 y_0\right]$$

$$= \frac{1}{19}\left[10 - 8 - \frac{(0.14989)(0.14989-1)}{2!}(18) + \frac{(0.14989)(0.14989-1)}{3!}(6)\right]$$

$$= \frac{1}{19}[2 + 1.146814 - 0.12742]$$

$$= \frac{1}{19}[3.01939]$$

$P_4 = 0.154$

$P_5 = 0.1542$

$$\boxed{P_3 = 0.1589}$$

$$P_4 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{P_3(P_3-1)}{2!}\Delta^2 y_0 - \frac{P_3(P_3-1)(P_3-2)}{3!}\Delta^3 y_0\right]$$

$$= \frac{1}{19}\left[10 - 8 - \frac{(0.1589)(0.1589-1)}{2!}(18) - \frac{(0.1589)(0.1589-1)}{3!}(6)\right]$$

$$= \frac{1}{19}[2 + 1.20286 + 0.4954079] = \frac{1}{19}[3.6982679]$$

$$\boxed{P_4 = 0.1541}$$

$$P_5 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{P_4(P_4-1)}{2!}\Delta^2 y_0 - \frac{P_4(P_4-1)(P_4-2)}{3!}\Delta^3 y_0\right]$$

$$= \frac{1}{19}\left[10 - 8 - \frac{(0.1541-1)(0.1541)}{2!}(18) - \frac{0.1541(0.1541-1)(0.1541-2)}{3!} \times 6\right]$$

$$= \frac{1}{19}[2 + 1.1732 - 0.24062]$$

$$\boxed{P_5 = 0.1543}$$

H.W The following value of $y = f(x)$ are given

x	10	15	20
y	1754	2648	2564

Find the value of x for $y = 3000$ by Iterative method

13/08/18 **Hermit's interpolating polynomial**

Given a set of data points (x_i, y_i, y_i') $i=0,1,..$ determine a polynomial of least degree, which is denoted by H_{2n+1} such that

$$\left.\begin{aligned} H_{2n+1}(x) &= y_i \\ &\vdots \\ H'_{2n+1}(x) &= y_i' \end{aligned}\right\} i=1,2\ldots \quad \text{—} (1)$$

This polynomial $H_{2n+1}(x)$ is Hermits interpolating polynomial.

Pb: Derive an interpolating polynomial in which both functions values and 1^{st} derivative values are to be assigned at each point of to the interpolating.

Soln:-

Given a set of data points. (x_i, y_i, y_i'), $i=0,1,\ldots n$ determine a polynomial of least, whic is denoted by $H_{2n+1}(x)$ such that,

$$H_{2n+1}(x_i) = y_i \text{ and}$$

$$H'_{2n+1}(x_i) = y_i', \quad i = 0, 1, \dots n$$

The polynomial $H_{2n+1}(x)$ is called Hermit's interpolation polynomial.

Since we have $2n+2$ conditions the number of coefficients to be determined is $2n+2$ and hence the degree of H_{2n+1}, x is $2n+1$.

The required polynomial by $H_{2n+1}(x)$ can be written as

$$H_{2n+1}(x) = \sum_{i=0}^{n} A_i(x)\, y_i + \sum_{i=0}^{n} B_i(x)\, y_i' \quad \text{——} (2)$$

where $A_i(x)$ and $B_i(x)$ are polynomial of degree $\leq 2n+1$ using (1) & (2) we obtain the following conditions.

(i) $A_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

(ii) $B_i(x_j) = 0$ for all i and j

(iii) $A_i'(x_j) = 0$ for all i and j $\Big\} \rightarrow (3)$

iv) $B_i'(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Since $A_i(x)$ and $B_i(x)$ are polynomials of degree $\leq 2n+1$ we write

$$A_i(x) = u_i(x)\, l_i^2(x) \text{ and}$$

$$B_i(x) = v_i(x)\, l_i^2(x)$$

$$\text{where } l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Note that $l_i(x)$ lagrange's interpolation polynomials and

$$l_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{—} (4)$$

Since $l_i^2(x)$ is a polynomial of degree $2n$ and

$l_i(x)$ and $B_i(x)$ are polynomial of degree $2n+1$ we see that $u_i(x)$ and $v_i(x)$ are polynomial of degree 1

Let $u_i(x) = a_i x + b_i$

$v_i(x) = c_i x + d_i$

$\therefore A_i(x) = (a_i x + b_i)\, l_i^2(x)$

$B_i(x) = (c_i x + d_i)\, l_i^2(x)$ — (5)

using the conditions (3) and (4) in (5) we obtain

$a_i x_i + b_i = 1$

$c_i x_i + d_i = 0$

$a_i + 2 l_i'(x_i) = 0$

$c_i = 1$

Hence we obtain $a_i = -2 l_i'(x_i)$

$b_i = 1 + 2 x_i l_i'(x_i)$

$c_i = 1$

and $d_i = -x_i$

Hence (5) becomes,

$$\boxed{\begin{aligned} A_i(x) &= [-2\, l_i'(x)\, x + 1 + 2x_i\, l_i'(x_i)]\, l_i^2(x) \\ &= [1 - 2(x - x_i)\, l_i'(x_i)]\, l_i^2(x) \\ \text{and } B_i(x) &= (x - x_i)\, l_i^2(x) \end{aligned}}$$

Thus the required Hermits interpolation polynomial is

$$\boxed{H_{2n+1}(x) = \sum_{i=0}^{n} A_i(x) y_i + \sum_{i=0}^{n} B_i(x) y_i'}$$

where

$A_i(x) = [1 - 2(x - x_i)\, l_i'(x_i)]\, l_i^2(x)$

$B_i(x) = (x - x_i)\, l_i^2(x)$.

8/1/18 P.L.

using Hermite's interpolation find $\sin 1.05$ for the following data.

x	1.0	1.1
$y = \sin x$	0.84147	0.89121
$y' = \cos x$	0.5403	0.45360

Here $n=1$, $x_0 = 1$, $x_1 = 1.1$

Hermite's interpolating polynomial is

$$\boxed{H_{2n+1}(x) = \sum_{i=0}^{n} A_i(x)\, y_i + \sum_{i=0}^{n} B_i(x)\, y_i}$$

here $n=1$, then

$$H_{2(1)+1}(x) = \sum_{i=0}^{1} A_i(x) y_i + \sum_{i=0}^{1} B_i(x) y_i$$

$$H_3(x) = \sum_{i=0}^{1} A_i(x) y_i + \sum_{i=0}^{1} B_i(x) y_i$$

where,

$$A_0(x) = [1 - 2(x-x_0)\, l_0'(x_0)]\, l_0^2(x)$$

$$A_1(x) = [1 - 2(x-x_1)\, l_1'(x_1)]\, l_1^2(x)$$

$$B_0(x) = (x-x_0)\, l_0^2(x)$$

$$B_1(x) = (x-x_1)\, l_1^2(x).$$

Now, $l_0 x = \dfrac{x-x_1}{x_0-x_1}$

$$= \frac{x-1.1}{1-1.1}$$

$$= \frac{x-1.1}{-0.1}$$

$$l_0 x = -10x + 11$$

$$l_0^2(x) = (-10x+11)^2 = 100x^2 - 220x + 121$$

$$l_0'(x) = -10 \text{ and } l_0'(x_0) = -10$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-1}{1.1-1} = 10x - 10$$

$$l_1'(x) = 10 \text{ and } l_1'(x_1) = 10$$

Now, $A_0(x) = [1 - 2(x-1)(-10)][100x^2 - 220x + 121]$

$$= (1 - 2x + 2)(-10)]\,[100x^2 - 220x + 121]$$

$$= [-10 + 20x - 20]\,(100x^2 - 220x + 121)$$

$$= [-30 + 20x](100x^2 - 220x + 121)$$

$$= -3000x^2 + 6600x - 3630 + 2000x^3 - 4400x^2 + 2420x$$

$$A_0(x) = 2000x^3 - 7400x^2 + 9020x - 3630$$

$\cancel{= -2000x^3 + 6300x^2 - 6600x + 2800}$

$= 2000x^3 - 6300x^2 + 6600x - 2299$

$A_1(x) = [1 - 2(x-1.1)(10)](100x^2 - 200x + 100)$

$= -2000x^3 + 6300x^2 - 6600x + 2300$

$B_0(x) = (x-1)(100x^2 - 220x + 121)$

$= 100x^3 - 320x^2 + 341x - 121.$

$B_1(x) = (x-1.1)(100x^2 - 200x + 100)$

$= 100x^3 - 310x^2 + 320x - 110$

$H_3(x) = A_0(x)y_0 + A_1(x)y_1 + B_0(x)y_0' + B_1(x)y_1'$

$= (2000x^3 - 6300x^2 + 6600x - 2299)(0.84147)$

$+ (-2000x^3 + 6300x^2 - 6600x + 2300)(0.89121)$

$+ (100x^3 - 320x^2 + 341x - 121)(0.5403)$

$+ (100x^3 - 310x^2 + 320x - 110)(0.45360)$

$= 1682.94x^3 - 5301.261x^2 + 5553.702x - 1934.53953$

$- 1782.42x^3 + 5614.623x^2 - 5881.986x + 2049.783$

$- 54.03x^3 - 172.896x^2 + 184.2423x - 65.3763$

$+ 45.36x^3 - 140.616x^2 + 145.152x - 49.896$

$H_3(x) = -0.09x^3 - 0.15x^2 + 1.1103x - 0.02883$

putting $x = 1.05$

$\boxed{H_3(x) = 0.8674237}$

20/08/18

unit - II

Numerical Differentiation and Integration

8.1 Derivatives using Newton's forward differentiation formula

Newton's interpolation formula for equal intervals is

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots \quad \text{—} (1)$$

where $p = \frac{x - x_0}{h}$ ——— ②

Differentiating eqn ① with respect to p we get.

$y(x) = y_0 + p\Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \ldots$

$$\frac{dy}{dp} = \Delta y_0 + \left(\frac{2p-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3p^2 - 6p + 2}{3!}\right)\Delta^3 y_0 + \left(\frac{4p^3 - 18p^2 + 22p - 6}{4!}\right)\Delta^4 y_0 + \cdots \quad ③$$

Differentiating eqn ② with respect to x, we have

$$\frac{dp}{dx} = \frac{1}{h}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

$$\left(\frac{dy}{dp}\right)_{x = x_0 + ph} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3p^2 - 6p + 2}{3!}\right)\Delta^3 y_0 + \left(\frac{4p^3 - 18p^2 + 22p - 6}{4!}\right)\Delta^4 y_0 + \cdots\right]$$

At $x = x_0$, $p = 0$

$$\left(\frac{dy}{dx}\right)_{x = x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{1}{h}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \left(\frac{6p^2 - 18p + 11}{12}\right)\Delta^4 y_0 + \cdots\right]$$

At $x = x_0$, $p = 0$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{x = x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

Derivatives of higher order can similarly be obtained.

Derivatives using Newton's backward difference formula

We know that Newton's interpolating formula for backward difference is

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots \quad (1)$$

where $p = \frac{x - x_n}{h}$

As before, differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2!}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{3!}\right)\nabla^3 y_n + \left(\frac{2p^3+9p^2+11p+3}{4!}\right)\nabla^4 y_n + \cdots\right]$$

at $x = x_n$, $p = 0$

$$\therefore \left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h}\left[\nabla y_n + (p+1)\nabla^3 y_n + \left(\frac{6p^2+18p+11}{12}\right)\nabla^4 y_n + \cdots\right]$$

At $x = x_n$, $p = 0$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

Similarly we can find the higher order derivatives.

24/08/18

8.3 Derivate using Stirling formula

The Stirling formula is

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-2)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \cdots \quad (1)$$

where $p = \frac{x - x_0}{h}$

as before differentiating ① with respect to x we get,

$$\frac{dy}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{(3p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right.$$

$$\left. + \frac{(4p^3-2p)}{4!}\Delta^4 y_{-2} + \cdots\right]$$

At $x = x_0$, $p = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\right.$$

$|||^{ly}$ we can derive

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right]$$

8·4 maxima and minima of the interpolation polynomial

Since the derivative of a function $y = f(x)$ given by a table of values is defined to be the derivative of the interpolation polynomial the maxima and minima of $f(x)$ can be contained by equation the first derivative to zero.

Newton's ~~inter~~ forward interpolation formula is,

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \quad \text{—①}$$

where $p = \frac{x - x_0}{h}$

$$\therefore \frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots$$

For y to be a maximum (or) minimum $\frac{dy}{dp} = 0$

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0 \quad \text{—②}$$

(neglecting higher order differences)

substitute in ⑥ the know values of Δy_0, $\Delta^2 y_0$ & $\Delta^3 y_0$ from the difference table we get a quadratic equation in p which can be solved for p.

The corresponding value of x at which $y(x)$ has maximum (or) minimum is given by $x = x_0 + ph$

Eg:(05/15 pl) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 51$ from the following data

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Soln:- Here $h = 10$

we have $p = \frac{x - x_0}{h} = \frac{51-50}{10} = 0.1$

At $x = 51$, $p = 0.1$

$\frac{p(p-1)}{2!}$ $p^2 - p$ $2p-1$ $\frac{p(p-1)(p-2)}{3!}$ $p(p^2-2p-p+2)$ $p(p^2-3p+2)$ p^3-3p^2+2p $3p^2-6p+2$

$$\left(\frac{dy}{dx}\right)_{x=51} = \left(\frac{dy}{dx}\right)_{p=0.1} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 y_0 + \frac{(4p^3-18p^2+22p-6)}{4!}\Delta^4 y_0 + \dots\right]$$

The difference table,

x	$p = \frac{x-50}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	0	19.96				
			16.69			
60	1	36.65		5.47		
			22.16		-9.23	
70	2	58.81		-3.76		11.99
			18.40		2.76	
80	3	77.21		-1.00		
			17.40			
90	4	94.61				

$$\left(\frac{dy}{dx}\right)_{p=0.1} = \frac{1}{10}\left[16.69 + \frac{(0.2-1)}{2}(5.47) + \left(\frac{3(0.1)^2 - 6(0.1)+2}{6}\right)(-9.23) + \left(\frac{4(0.1)^3 - 18(0.1)^2 + 22(0.1) - 6}{24}\right)(11.99)\right] \dots$$

$= \frac{1}{10}[16.69 - 2.188 - 2.1998 - 1.9863]$

$\boxed{\frac{dy}{dx} = 1.0316}$ //

$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{(6p^2-18p+11)}{12}\Delta^4 y_0 + \cdots\right]$

$= \frac{1}{100}\left[5.47 + (0.1-1)(-9.23) + \frac{6(1)^2 - 18(1) + 11)}{12} \times 11.99\right]$

$= \frac{1}{100}[5.47 + 8.307 + 9.2523]$

$\boxed{\frac{d^2y}{dx^2} = 0.2303}$ //.

pb Find $y'(x)$ given

x	0	1	2	3	4
$y(x)$	1	1	15	40	85

Hence find $y'(x)$ at $x = 0.5$

Soln:-

Here $h = 1$

Newton's forward interpolation formula,

$y_p' = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 y_0 + \frac{(4p^3-18p^2+22p-6)}{4!}\Delta^4 y_0 + \cdots\right]$

where $p = \frac{x - x_0}{h}$

$p = x$ at $x_0 = 0$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		0			
1	1		14		
		14		-3	
2	15		11		12
		25		9	
3	40		20		
		45			
4	85				

$\frac{1}{h} = \frac{1}{1} = 1.$

$\therefore y_p' = \Delta y_0 + \frac{(2x-1)}{2}\Delta^2 y_0 + \frac{3x^2-6x+2}{6}\Delta^3 y_0 + \frac{4x^3 - 18x^2 + 22x - 6}{24}\Delta^4 y_0$

$= 0 + \frac{(2x-1)}{2}(14) + \frac{(3x^2-6x+2)}{6}(-3) + \frac{(4x^3-18x^2+22x-6)}{24}$

$= 7(2x-1) - \frac{(3x^2-6x+2)}{2} + (2x^3-9x^2+11x-3)$

$y'(x) = 2x^3 - \frac{21}{2}x^2 + 28x - 11$

Now, y' at $x = 0.5$

Then $y'(0.5) = 2(0.5)^3 - \frac{21}{2}(0.5)^2 + 28(0.5) - 11$

$= 0.625$ //

04/09/18 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 89$

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Soln:- here $h = 10$

Newton's backward formula, $p = \frac{x - x_n}{h} = \frac{89-90}{10} = \frac{-1}{10}$

$\boxed{p = -0.1}$

The difference table

x	$p = \frac{x-50}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	0	19.96	16.69	5.47	-9.23	11.99
60	1	36.65	22.16	-3.76	2.76	
70	2	58.81	18.40	-1.00		
80	3	77.21	17.40			
90	4	94.61				

$$\left(\frac{dy}{dx}\right)_{x=89} = \left(\frac{dy}{dx}\right)_{-0.1} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2!}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{3!}\right)\nabla^3 y_n + \left(\frac{2p^3+9p^2+11p+3}{4!}\right)\nabla^4 y_n\right]$$

$$= \frac{1}{10}\left[(17.40)+\left(\frac{2(-0.1)+1}{2!}\right)(-1.00)+\frac{3(-0.1)^2+6(-0.1)+2}{3!}(2.76)\right.$$
$$\left.+\left(\frac{2(-0.1)^3+9(-0.1)^2+11(-0.1)+2}{4!}\right)11.99\right]$$

$$= \frac{1}{10}\left[17.40 - 0.9 + 0.6578 + 0.9932\right]$$

$$= \frac{1}{10}\left[18.151\right] = 1.8151$$

$$\boxed{\frac{dy}{dx} = 1.8151} \quad //$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n\right]$$

$$= \frac{1}{(0.1)^2}\left[(-1.00)+(2.76)+\overset{0.9167}{\frac{11}{12}}(11.99)\right]$$

$$= \frac{1}{100}\left[12.751233\right]$$

$$\boxed{\frac{d^2y}{dx^2} = 0.1275}$$

20/09/18

unit – III

Numerical integration

Newton's – Cote's quadrature formula

Let $I = \int_a^b f(x)\,dx$ where $f(x)$ taken the values $y_0, y_1, y_2 \cdots y_n$ for $x = x_0, x_1, x_2, \cdots x_n$.

Let us divide the intervals (a, b) into n sub intervels of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $\cdots\cdots x_n = x_0 + nh$

Now,

$$I = \int_a^b f(x)\,dx$$

$x = x_0 + ph$
differentiating with respect to p
$\frac{dx}{dp} = 0 + 1(h)$
$\frac{dx}{dp} = h$

$p = \frac{x - x_0}{h}$
$ph = x - x_0$
$\therefore x = x_0 + ph$.

$$= h\int_0^n f(x_0 + ph)\,dp \quad \text{where } p = \frac{x - x_0}{h}$$

$$= h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right] dp$$

[$\because$ by newton's forward difference formula]

$$= h\left[y_0 p + \frac{p^2}{2}\cdot\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - \frac{2p^3}{2}\ 2\frac{p^2}{2}\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$= h\left[n\cdot y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\right.$$

$$\therefore \int_a^b f(x)dx = h\left[n\cdot y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots\right]$$

Trapezeridal Rule

We know that By Newton's cote's quadrature formula we have

$$\int_{a=x_0}^{b=x_n} f(x)dx = h\left[n\cdot y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 \cdots\right]$$

put $n=1$

$$\int_{x_0}^{x_1} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] \rightarrow \text{difference table} \quad \begin{matrix} y_0 & \\ & y_1 - y_0 \\ y_1 & \\ y_2 & \end{matrix}$$

$$= \frac{h}{2}\left[2y_0 + y_1 - y_0\right]$$

$$\boxed{\therefore \int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(y_0 + y_1)}$$

IIIly,

$$\int_{x_1}^{x_2} f(x)dx = \frac{h}{2}\left[y_1 + y_2\right]$$

$$\int_{x_2}^{x_3} f(x)dx = \frac{h}{2}\left[y_2 + y_3\right]$$

. . . .

. . . .

$$\int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Adding these equation, we have

$$\int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2y_1 + 2y_2 + \cdots + 2y_{n-1}\right]$$

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$$\therefore \int_{x_0=a}^{x_n=b} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

which is a required Trapezeridal Rule.

Simpson's $\frac{1}{3}$ Rule

We know that By Newton's cote's quadrature formula, we have

$$\int_{a=x_0}^{b=x_n} f(x)\,dx = h\left[n\cdot y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{\Delta^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots\right]$$

Put $n=2$

$$\therefore \int_{x_0}^{x_2} f(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{2} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right]$$

$$= \frac{h}{3}\left[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[\cancel{6y_0} + 6y_1 - \cancel{6y_0} + y_2 - 2y_1 + y_0\right]$$

$$= \frac{h}{3}\left[4y_1 + y_2 + y_0\right]$$

$$\boxed{\therefore \int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]}$$

|||ly

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\int_{x_4}^{x_6} f(x)\,dx = \frac{h}{3}\left[y_4 + 4y_5 + y_6\right]$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Adding these equation, we get,

$$\int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

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$$\therefore \int_{x_0=a}^{x_n=b} f(x)dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\dots+y_{n-1})+2(y_2+y_4+\dots+y_{n-2})\right]$$

which is the required Simpson's 1/3 rule.

Simpson's 3/8 Rule :-

we know that By Newton's cote's quadrature formula, we have

$$\int_{a=x_0}^{b=x_n} f(x)dx = h\left[n\cdot y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0 + \dots\right]$$

put $n=3$

$$\int_{x_0}^{x_3} f(x)dx = h\left[3\cdot y_0 + \frac{3^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{3^3}{3}-\frac{3^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{3^4}{4}-3^3+3^2\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(9-\frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4}-18\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{18-9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81-72}{4}\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{\cancel{6}_2}\left(\frac{\cancel{9}^3}{4}\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \frac{3}{8}\Delta^3 y_0\right]$$

$$= \frac{8h}{8}\left[3y_0 + \frac{9}{2}(y_1-y_0) + \frac{9}{4}(y_2-2y_1+y_0) + \frac{3}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{h}{8}\left[24y_0 + 36(y_1-y_0) + 18(y_2-2y_1+y_0) + 3(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{h}{8}\left[24y_0 + \cancel{36y_1} - 36y_0 + 18y_2 - \cancel{36y_1} + 18y_0 + 3y_3 - 9y_2 + 9y_1 - 3y_0\right]$$

$$= \frac{h}{8}\left[3y_0 + 9y_1 + 9y_2 + 3y_3\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

$$\boxed{\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[y_0+3y_1+3y_2+y_3\right]}$$

$|||^{ly}$

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}\left[y_3+3y_4+3y_5+y_6\right]$$

$$\int_{x_6}^{x_9} f(x)dx = \frac{3h}{8}\left[y_6+3y_7+3y_8+y_9\right]$$

. . . .

. . . .

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Adding these equation we get,

$$\int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \int_{x_6}^{x_9} f(x)dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)\,dx =$$

$$\frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\right]$$

which is the required simpson's 3/8 rule.

Weddle's rule

put $n=6$ is newton's - cote's quadrature formula, we get,

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

|||ly

$$\int_{x_6}^{x_{12}} f(x)\,dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

$$\int_{x_{12}}^{x_{18}} f(x)\,dx = \frac{3h}{10}\left[y_{12} + 5y_{13} + y_{14} + 6y_{15} + y_{16} + 5y_{17} + y_{18}\right]$$

$\vdots$

$$\int_{x_{n-6}}^{x_n} f(x)\,dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$$

Adding we get,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left[(y_0+5y_1+y_2+6y_3+y_4+5y_5) + (2y_6+5y_7+y_8+6y_9 + y_{10}+5y_{11}) + \cdots + (32y_{n-4} + 12y_{n-3} + 32y_{n-2} + 14y_{n-1} + y_n)\right]$$

which is the required weddle's rule.

Boole's rule :-

putting $n=4$ in newton - cote's quadrature formula, we obtain,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\Big[(7y_0+32y_1+12y_2+32y_3+14y_4$$

$$+32y_5+12y_6+32y_7+14y_8)+\cdots\cdots$$

$$+(32y_{n-4}+12y_{n-3}+32y_{n-2}+14y_{n-1}+y_n)\Big]$$

This is known as Boole's rule.

Pb Evaluate $\int_0^5 \frac{dx}{4x+5}$ by (i) Trapezoidal test (ii) simpson's $\frac{1}{3}$ rule (iii) simpson's $\frac{3}{8}$ rule (iv) weddle's rule.

Soln:- Take $n=10$

$$\therefore h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5$$

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$y=\frac{1}{4x+5}$	0.2 y_0	0.14 y_1	0.11 y_2	0.09 y_3	0.08 y_4	0.07 y_5	0.06 y_6	0.05 y_7	0.04 y_8	0.04 y_9	0.04 y_{10}

(i) Trapezoidal test where $h=0.5$

$$\int_a^b f(x)\,dx = \frac{h}{2}\Big[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\Big]$$

$$= \frac{0.5}{2}\Big[(y_0+y_{10})+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\Big]$$

$$= \frac{0.5}{2}\Big[(0.2+0.04)+2(0.14+0.11+0.09+0.08+0.07+0.06+0.05+0.04+0.04)\Big]$$

$$= \frac{0.5}{2}\Big[0.24+2(0.68)\Big]$$

$= \frac{0.5}{2}\left[1.6\right] = (0.25)(1.6)$

$= 0.4$ //

(ii) Simpson's 1/3 rule :-

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1})+2(y_2+y_4+\cdots+y_{n-2})\right]$$

$$= \frac{0.5}{3}\left[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{0.5}{3}\left[(0.2+0.04)+4(0.14+0.09+0.07+0.05+0.04)+ 2(0.11+0.08+0.06+0.04)\right]$$

$$= \frac{0.5}{3}\left[(0.24)+4(0.39)+2(0.29)\right]$$

$$= \frac{0.5}{3}\left[(0.24)+1.56+0.58\right] = \frac{0.5}{3}\left[2.38\right] = (0.167)(2.38)$$

$= 0.39746$

(iii) Simpson's 3/8 rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+ 2(y_3+y_6+y\cdots+y_{n-3})\right]$$

$$= \frac{3h}{8}\left[(y_0+y_{10})+3(y_1+y_2+y_4+y_5+y_7+y_8)+ 2(y_3+y_6+y_9)\right]$$

$$= \frac{3(0.5)}{8}\Big[$$

Pb Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with $h=0.2$

Hence determine the value of π.

Soln:- here $h=0.2$

$$y = \frac{1}{1+x^2}$$

x	0	0.2	0.4	0.6	0.8	1.
$y=\frac{1}{1+x^2}$	1	0.9615	0.8621	0.7353	0.6098	0.5

By Trapezoidal rule,

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0+y_5)+(y_1+y_2+y_3+y_4)\right]$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{0.2}{2}\left[(1+0.5)+2(0.9615+0.8621+0.7353+0.6098)\right]$$

$$= 0.7837 \quad \text{--- (1)}$$

To find the value of π

By autual integration

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} \quad \text{--- (2)}$$

from (1) & (2) we get

$$\frac{\pi}{4} = 0.7837$$

$$\boxed{\pi = 3.318}$$

Pb Evaluate $\int_0^1 e^{-x^2}dx$ by dividing the range into 4 equa[l] parts using Trapezoidal rule.

Soln:-

$$y = e^{-x^2}$$

Take $h = 0.25$

x	0	0.25	0.5	0.75	1
$y=e^{-x^2}dx$	1	0.9394	0.7788	0.5698	0.3679

By Trapezoidal rule,

$$\int e^{-x^2}dx = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

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$$= \frac{0.25}{2}\left[1.3679 + 2(2.288)\right]$$

$$= 0.7430$$

Pb Evaluate $\int_0^{\pi/2} \sin x\, dx$ by simpson's $\frac{1}{3}$ rule dividing the range into six equal parts.

Soln:- $y = \sin x\, dx$

we subdivide this interval into six equal parts with

$h = \frac{\pi}{12}$

x	0	π	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$y = \sin x$	$\cdot 0$	0.2588	0.500	0.7071	0.8660	0.9659	1.0000

by simpson's $\frac{1}{3}$ rule,

$$\int_0^{\pi/2} \sin x\, dx = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$= \frac{\pi}{36}\left[(0+1) + 2(0.5 + 0.866) + 4(0.2588 + 0.7071 + 0.9659)\right]$$

$$= 0.0873\left[1 + 2(1.366) + 4(1.9318)\right]$$

$$= 1.0004 \;//$$

Pb Calculate $\int_{0.5}^{0.7} e^{-x} x^{1/2}\, dx$ taking 5 ordinates by simpson's $\frac{1}{3}$ rule.

Soln:- $y = e^{-x} x^{1/2}$ length of the interval is 0.2.

Take $h = 0.04$

x	0.5	0.54	0.58	0.62	0.66	0.7
$y = e^{-x} x^{1/2}$	0.4289	0.4282	0.4264	0.4236	0.4199	0.4155

Simpson's $\frac{1}{3}$ rule is,

$$\int_{0.5}^{0.7} e^{-x} x^{1/2}\, dx = \frac{h}{3}\left[(y_0 + y_5) + 2(y_2 + y_4) + 4(y_1 + y_3)\right]$$

$$= \frac{0.04}{3}\left[(0.4289 + 0.4155) + 2(0.4264 + 0.4199) + 4(0.4282 + 0.4236)\right]$$

$$= 0.0793 \;//$$

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Pb Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3}\,dx$ using Simpson's $\frac{1}{3}$ rule with $h=0.25$.

Soln:- $y = \frac{x^2}{1+x^3}$

x	0	0.25	0.5	0.75	1
$y=\frac{x^2}{1+x^3}$	0	0.0615	0.2222	0.3956	0.5

By Simpson's $\frac{1}{3}$ rule,

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{h}{3}\left[(y_0+y_4)+2y_2+4(y_1+y_3)\right]$$

$$= \frac{0.25}{3}\left[0.5+2(0.2222)+4(0.0615+0.3956)\right]$$

$$= 0.2311$$

By actual integration,

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3}\,dx$$

$$= \frac{1}{3}\left[\log(1+x^3)\right]_0^1$$

$$= \frac{1}{3}\log(2)$$

$$= \log 2^{1/3}$$

$\therefore \log 2^{1/3} = 0.2311$

Pb Evaluate $\int_0^{10} \frac{dx}{1+x^2}$ by using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$ rule.

Soln:- Here length of interval is 10.

Take $h=1$

$y = \frac{1}{1+x^2}$

x	0	1	2	3	4	5	6	7	8	9	10
$y=\frac{1}{1+x^2}$	0	0.5	0.2	0.1	0.0588	0.0385	0.0270	0.02	0.0154	0.0122	0.0099

(i) Trapezoidal rule

$$\int_0^{10} \frac{dx}{1+x^2} = \frac{1}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\right]$$

$= \frac{1}{2}\left[(1+0.0099)+2(0.5+0.2+0.1+0.0588+0.0385+0.0270+0.02+0.0154+0.0122)\right]$

$=$

(ii) Simpson's 1/3 rule,

$$\int_0^{10} \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_{10})+2(y_2+y_4+y_6+y_8)+4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{1}{3}\left[(1+0.0099)+2(0.2+0.0588+0.027+0.0154)+4(0.5+0.1+0.0385+0.02+0.0122)\right]$$

$$= \frac{1}{3}(4.2951)$$

$$= 1.4317$$

Pb The velocity v of a particle at distance S from a point on its path is given by the table below.

S in meters	0	10	20	30	40	50	60
v in m/sec	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by using Simpson's 1/3 rule.

Soln:- Here h=10

w.k.T $v = \frac{ds}{dt}$ ~~Hea~~

$dt = \frac{ds}{v}$

To find the time taken to travel 60 meters, we have to evaluate $\int_0^{60} dt = \int_0^{60} \frac{ds}{v}$

Let $y = \frac{1}{v}$ the table values for y for different values of s are given below

S	0	10	20	30	40	50	60
$y=\frac{1}{v}$	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263

Simpson's 1/3 rule:

$$\int_0^{60} y\,ds = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$\int_0^{60} dt = \frac{10}{3}\Big[(0.0213 + 0.0263) + 2(0.0156 + 0.0164) + 4(0.0172 + 0.0154 + 0.0192)\Big]$$

$$= 1.0627$$

$\therefore$ Time taken to travel 60 meters $= 1.0627$ Seconds.

Pb A curve passes through the points as given in the table, find,

(i) The area bounded by the curves the x-axis, $x=1$ and $x=9$.

(ii) The value of the solid generated by revolving this area about the x-axis.

x	1	2	3	4	5	6	7	8	9
y	0.2	0.7	1	1.3	1.5	1.7	1.9	2.1	2.3

soln:-

(i) Here $h=1$ $A = \int_1^9 y\, dx$

Simpson's 1/3 rule;

$$\int_1^9 y\, dx = \frac{h}{3}\Big[(y_0+y_9) + 2(y_2+y_4+y_6) + 4(y_1+y_3+y_5+y_7)\Big]$$

$$= \frac{1}{3}\Big[(0.2+2.3) + 2(1+1.5+1.9) + 4(0.7+1.3+1.7+2.1)\Big]$$

$$= 11.5 \text{ sq. units.}$$

$\therefore$ The required Area $= 11.5$ sq. units

volume $V = \pi \int_1^9 y^2 dx$

we find $\int_1^9 y^2 dx$ using simpson's 1/3 rule

$$\therefore \int_1^9 y^2 dx = \frac{1}{3}\Big[(0.2^2 + 2.3^2) + 2(1^2 + 1.5^2 + 1.9^2) + 4(0.7^2 + 1.3^2 + 1.7^2 + 2.1^2)\Big]$$

$$= \frac{1}{3}[5.33 + 13.72 + 37.92]$$

$$= \frac{1}{3}[56.97]$$

$$= 18.99$$

$\therefore$ The required volume $V = \pi(18.99)$

$$= 59.6588 \text{ cubic units.}$$

Pb Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Romberg's method correct to 4 decimal places. Hence deduce an approximate value.

Soln:- Let $y = \frac{1}{1+x^2}$ and let $I = \int_0^1 \frac{dx}{1+x^2}$

Take $h = 0.5$ the tabulated values of y are

x	0	0.5	1
$y = \frac{1}{1+x^2}$	1	0.8	0.5

using trapezoidal rule,

$$I_1 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_2) + 2y_1\right]$$

$$= \frac{0.5}{2}\left[1 + 0.5) + 1.6\right]$$

$$= 0.775$$

Take $h = 0.25$ the tabulated values of y are,

x	0	0.25	0.50	0.75	1.00
$y = \frac{1}{1+x^2}$	1	0.9412	0.80	0.64	0.5

using Trapezoidal rule,

$$I_2 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.25}{2}\left[(1 + 0.5) + 2(0.9412 + 0.80 + 0.64)\right]$$

$$= 0.7828$$

Take $h = 0.125$ the tabulated values of y are

x	0	0.125	0.25	0.375	0.50	0.625	0.750	0.875	1
$y = \frac{1}{1+x^2}$	1	0.9846	0.9412	0.8767	0.80	0.7191	0.64	0.5664	0.5

using Trapeziodal rule,

$$I_3 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + \cdots + y_7)\right]$$

$$= \frac{0.125}{2}\left[(1 + 0.5) + 2(0.9846 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664\right]$$

$$= (0.0625)\left[1.5 + 2(5.528)\right]$$

$$= 0.78475$$

using Romberg's formula for I_1 and I_2 we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$= 0.7828 + \left(\frac{0.7828 - 0.775}{3}\right)$$

$$= 0.7828 + 0.0026$$

$$= 0.7854$$

using Romberg's formula for I_2 and I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right)$$

$$= 0.78475 + \left(\frac{0.78475 - 0.7828}{3}\right)$$

$$= 0.78475 + 0.00065$$

$$= 0.7854$$

$$\therefore I = \int_0^1 \frac{dx}{1+x^2} = 0.7854 \quad \text{——} \; (1)$$

By actual evaluation of the definite integral we have

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4} \quad \text{——} \; (2)$$

from (1) and (2) we have $\frac{\pi}{4} = 0.7854$

$\therefore$ Hence $\pi \simeq 3.1416$

11) Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method. Hence obtain an approximate value for π.

Soln:-

$$y = \frac{1}{x^2+4} \qquad \text{let } I = \int_0^2 \frac{dx}{x^2+4}$$

Take $h = 1$

x	0	1	2
y	0.25	0.20	5

using Trapezoidal rule,

$$I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0 + y_1) + 2y_1\right]$$

$$= 0.5\left[(0.25 + 0.125) + 2(0.20)\right]$$

$$= 0.3875$$

Take $h = 0.5$ the tabulated values of y are

x	0	0.5	1.0	1.5	2
y	0.25	0.2353	0.20	0.160	0.125

using Trapezoidal rule,

$$I_2 = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$= 0.25\left[(0.25+0.125)+2(0.2353+0.2+0.16)\right]$$

$$= 0.3914$$

Take $h = 0.25$ the tabulated values of y are,

x	0	0.25	0.50	0.75	1.0	1.25	1.50	1.75	2.0
y	0.25	0.2462	0.2353	0.2192	0.20	0.1798	0.160	0.1416	0.125

By Trapezoidal rule,

$$I_3 = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= \left(\frac{0.25}{2}\right)\left[(0.25+0.125)+2(0.2462+0.2353+0.2192+0.20+0.1798+0.16+0.1416)\right]$$

$$= (0.125)(3.1392)$$

$$= 0.3924$$

using Romberg's formula for I_1 and I_2 we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$= 0.3914 + \left(\frac{0.3914 - 0.3875}{3}\right)$$

$$= 0.3953 \quad \text{——} (1)$$

using Romberg's formula for I_2 and I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right)$$

$$= 0.3924 + \left(\frac{0.3924 - 0.3914}{3}\right)$$

$$= 0.3927 \quad \text{——} (2)$$

Since (1) and (2) are not equal we go for one more application of Trapezoidal rule taking $h = 0.125$

Take $h = 0.125$ the tabulated values are

x	0	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.00
y	0.25	0.249	0.2462	0.2415	0.2353	0.2278	0.2192	0.2098	0.20

1.125	1.250	1.375	1.500	1.625	1.750	1.875	2.000
0.1899	0.1798	0.1698	0.160	0.1506	0.1416	0.1331	0.125

By Trapezoidal rule,

$$I_4 = \frac{h}{2}\left[(y_0+y_{16}) + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9+y_{10}+y_{11}+y_{12}+y_{13}+y_{14}+y_{15})\right]$$

$$= \left(\frac{0.125}{2}\right)\left[(0.25+0.125)+2(0.2949+0.2462+0.2415+0.2353+0.2278+0.2192+0.2098+0.20 + 0.1899+0.1798+0.1698+0.160+0.1506+0.1416+0.1337)\right]$$

$I_4 = 0.3926$

using Romberg's formula for I_3 and I_4 we have

$$I = I_4 + \left(\frac{I_4 - I_3}{3}\right)$$

$$= 0.3926 + \left(\frac{0.3926 - 0.3924}{3}\right)$$

$I = 0.3927$

Since ② & ③ are almost equal we can take

$$I = \int_0^2 \frac{dx}{x^2+4} = 0.3927 \quad \text{——} (A)$$

By actual integration

$$\int_0^2 \frac{dx}{x^2+4} = \int_0^2 \frac{dx}{x^2+2^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \frac{1}{2}\left[\frac{\pi}{4}\right] = \frac{\pi}{8} \quad \text{——} (B)$$

$\therefore$ from (A) and (B) we get $\frac{\pi}{8} = 0.3927$

$\therefore \pi \simeq 3.1416$

12) Evaluate $\int_0^1 \frac{dx}{1+x}$ using (i) Trapezoidal rule (ii) Simpson's one third rule (iii) Simpson's 3/8 rule (iv) Weddle's rule (v) find the error in each method by comparing with the actual integration upto 4 places of decimals. Take $h = 1/6$ for all cases.

Soln:- here $h = 1/6$

$$y = f(x) = \frac{1}{1+x}$$

x	0	1/6	2/6	3/6	4/6	5/6	1
$y = \frac{1}{1+x}$	1	0.8571	0.7571	0.667	0.60	0.5455	0.5

(i) Trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$\simeq \frac{1}{2}\left[(\cancel{1+0.5}\ 1+0.5)+2(0.857+0.75+0.6667+0.5455)\right]$$

$$= 0.6932$$

(ii) Simpson's 1/3 rule,

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$\simeq \frac{1}{18}\left[(1+0.5)+2(0.75+0.6)+4(0.8571+0.6667+0.5455)\right]$$

$$= 0.6932$$

(iii) Simpson's 3/8 rule

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$\simeq \frac{1}{16}\left[(1+0.5)+3(0.8571+0.75+0.6+0.5455)+2(0.6667)\right]$$

$$= 0.6932$$

(iv) weddley's rule,

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$\simeq \frac{1}{20}\left[1+5(0.8571)+0.75+6(0.6667)+0.6+5(0.5455)+0.5\right]$$

$$= 0.69320$$

v) By actual integration,

$$\int_0^1 \frac{dx}{1+x} = \left[\log_e(1+x)\right]_0^1 = \log_e 2 = 0.6931$$

comparing (i) and (v) error in trapezoidal rule is $0.6931 - 0.6949 = -0.0018$

Comparing (ii) and (v) error in Simpson's 1/3 rule is

$0.6931 - 0.6932 = -0.0001$

comparing (iii) and (v) error in simpson's 3/8 rule

$0.6931 - 0.6932 = 0.0001$

comparing (iv) and (v) error in weddlay's rule

$0.6931 - 0.6932 = -0.0001$

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8.6 Gaussian Quadrature formula

The formula that ~~we have~~

Two point Gaussian Quadrature formullae

Consider the integral

$$I = \int_{-1}^{1} f(x)\,dx$$

Let $I = a_1 f_1(x) + a_2 f_2(x)$ —— ①

where the coefficients a_1, a_2 and the functions arguments x_1, x_2 are to be determined.

To determine the four unknowns a_1, a_2, x_1, x_2 we require four conditions. For this purpose we impose the conditions ~~that~~ for this equation ① is valid for any polynomial of degree three or less.

In particular ① is true if $f(x) = x^3$, $f(x) = x^2$, $f(x) = x$ and $f(x) = 1$

$f(x) = x^3$ gives $a_1 x_1^3 + a_2 x_2^3 = \int_{-1}^{1} x^3 dx$

ie) $a_1 x_1^3 + a_2 x_2^3 = 0$ —— ②

lllly $f(x) = x^2$ gives $a_1 x_1^2 + a_2 x_2^2 = 2/3$ —— ③

$f(x) = x$ gives $a_1 x_1 + a_2 x_2 = 0$ —— ④

$f(x) = 1$ gives $a_1 + a_2 = 2$ —— ⑤

Multiplying ④ by x_1^2 and subtracting from ② we get,

$$a_2 (x_2^3 - x_2 x_1^2) = 0$$

$$\therefore a_2 x_2 (x_2^2 - x_1^2) = 0$$

ie) $a_2 x_2 (x_2 + x_1)(x_2 - x_1) = 0$

$\therefore$ Either $a_2 = 0$ (or) $x_2 = 0$ (or) $x_1 = x_2$ (or) $x_1 = -x_2$.

The cases $a_2 = 0$, $x_2 = 0$ and $x_1 = x_2$ give rise to invalid equations and hence we choose $x_1 = -x_2$.

$\therefore$ equation ② becomes

$$a_1 - a_2 = 0$$

from ⑤ and ⑥ we get $a_1 = a_2 = 1$

Now from ③ we get $x_1^2 + x_1^2 = 2/3$ and 1 hence

$$I = \int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right)$$

$$= f(0.5773) + f(-0.5773)$$

This is known as Gauss two point quadrature formula. Thus by adding two values of the function $f(x)$ we get and approximate value of the integral and the formula gives the exact value if $f(x)$ is any polynomial of degree 3 or less.

Remark:-1

In deriving the Gaussian two point quadrature formula we have assumed that the integration is from -1 to 1 which simplified the mathematical calculation.

If the limit is from a to b, then we shall apply a suitable change of variable to bring the integration from -1 to 1. we replace the given variable x by another variable t which are related by the following formula

$$x = \frac{(b-a)t + (b+a)}{2}$$

Clearly when $x=a$, $t=-1$ and when $x=b$, $t=1$ and

$$dx = \left(\frac{b-a}{2}\right) dt$$

$$\therefore \int_{a}^{b} f(x)\,dx = \left(\frac{b-a}{2}\right) \int_{-1}^{1} f\left[\frac{(b-a)t + (b+a)}{2}\right] dt$$

Remark 2:-

Gaussian two point quadrature formula requires only two functional evaluations and gives a good estimate of the value of the integral.

Pb Evaluate $\int_{0}^{1} \frac{dx}{1+x^2}$ by two and three point Gaussian quadrature formula and hence find the value of π.

Soln:- Let $f(x) = \frac{1}{1+x^2}$

Here $a=0$ and $b=1$

To change the limit of the integration from -1 to 1 put,

$$x = \frac{(b-a)t + b+a}{2} = \frac{t+1}{2}$$

$$\therefore I = \int_0^1 \frac{dx}{1+x^2} = \frac{1}{2}\int_{-1}^{1} f\left(\frac{t+1}{2}\right) dt$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{1}{1+\left(\frac{t+1}{2}\right)^2} dt = 2\int_{-1}^{1} \frac{dt}{t^2+2t+5}$$

$$= \int_{-1}^{1} g(t)\, dt \quad \text{where } g(t) = \frac{2}{t^2+2t+5}$$

By Gauss two point quadrature formula we have

$$I = \int_{-1}^{1} g(t)\, dt = g\left(\frac{1}{\sqrt{3}}\right) + g\left(\frac{-1}{\sqrt{3}}\right)$$

$$\therefore I = 2\left[\left\{\frac{1}{\frac{1}{3}+\frac{2}{\sqrt{3}}+5}\right\} + \frac{1}{\frac{1}{3}-\frac{2}{\sqrt{3}}+5}\right]$$

$$\therefore I = 0.7868 \quad \text{--- (1)}$$

By actual integration

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \pi/4 \quad \text{--- (2)}$$

$\therefore$ from (1) & (2) we have

$$\frac{\pi}{4} = 0.7868$$

$$\boxed{\therefore \pi = 3.1472}$$

Gaussian three point formula is given by

$$I = 0.55555555\, g(-0.77459667) + 0.88888889\, g(0)$$
$$+ 0.55555555\, g(0.77459667)$$
$$= 0.274293787 + 0.355555548 + 0.155417688$$
$$= 0.785267023$$

The corresponding approximate value of π is given by

$$\pi = 4(0.785267023)$$

$$\therefore \pi = 3.141068092$$

Q) find $\int_0^{\pi/2} \sin x\, dx$ by two and three point Gaussian qudature formula.

Soln:- Here $f(x) = \sin x$, $a = 0$ and $b = \pi/2$

To change the limit of the integration format -1 to 1 put

$$x = \frac{(b-a)t + (b+a)}{2} = \frac{\pi}{4}(t+1)$$

$$\therefore I = \int_0^{\pi/2} \sin x \, dx = \frac{\pi}{4}\int_{-1}^{1} f\left(\frac{\pi}{4}(t+1)\right) dt$$

$$= \pi/4 \int_{-1}^{1} \sin\left[\frac{\pi}{4}(t+1)\right] dt$$

$$= \int_{-1}^{1} g(t)\, dt \quad \text{where } dt = \frac{\pi}{4}\sin\left[\frac{\pi}{4}(t+1)\right]$$

By Gaussian two point qudature formula we have

$$I = \int_{-1}^{1} g(t)\, dt = g(0.5773) + g(-0.5773)$$

$$\therefore I = \frac{\pi}{4}\sin\left[\frac{1.5773\,\pi}{4}\right] + \frac{\pi}{4}\sin\left[\frac{0.4227\,\pi}{4}\right]$$

$$= \frac{\pi}{4}(1.2713)$$

$$= 0.9985$$

Gaussian three point formula is given by

$$I = 0.55555555\, g(-0.77459667) + 0.88888889\, g(0)$$
$$+ 0.55555555\, g(0.77459667)$$
$$= 0.076841659 + 0.49365366 + 0.429512797$$
$$= 1.000008116$$

we note that the actual value of the integral is 1 and that gaussian qudrature formulae provide a good approximation.

Q. Evaluate $I = \int_{-1}^{1} e^{-x^2}\cos x \, dx$ by Gauss two and three point qudrature formula.

Soln:- Gauss two point quadrature formula is,

$$I = \int_{-1}^{1} f(x)\, dx = f(0.5773) + f(-0.5773)$$

Here $f(x) = e^{-x^2}\cos x$

$$\therefore I = 0.716536528 + 0.716536528$$
$$= 1.433073056$$

Gauss three point qudrature formula is

$I = 0.55555555 + f(-0.77459667) + 0.88888889 f(0)$

$+ 0.55555555 + (0.77459667)$

$= 0.304867487 + 0.88888889 + 0.30486787$

$= 1.498623865$

8.7 Numerical Evaluation of Double integrals :.

If $f(x,y)$ is a continuous function defined on a closed rectangle

$$R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$$

then $\iint_R f(x,y)\,dx\,dy$ can be expressed as

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \quad (\text{or}) \quad \int_a^b \int_c^d f(x,y)\,dy\,dx$$

In this section we extend trapezoidal rule and Simpson's rule for numerical integration of double integrals in which the limits of the integrals are constants.

Trapezoidal rule for double integrals :-

Consider,

$$I = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(x,y)\,dx\,dy$$

where $x_{i+1} = x_i + h$ and $y_{j+1} = y_j + k$

By applying trapezoidal rule to inner integral, we get

$$I = \frac{h}{2} \int_{y_j}^{y_{j+1}} [f(x_i, y) + f(x_{i+1}, y)]\,dy$$

again applying trapezoidal rule, we have

$$I = \frac{hk}{4} [f(x_i, y_j) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f_{i+1, j+1}]$$

$$\therefore I = \frac{hk}{4} [f_{i,j} + f_{i+1,j} + f_{i,j+1} + f_{i+1,j+1}] \quad \text{--- (1)}$$

where $f_{i,j} = f(x_i, y_j)$

To evaluate

$$I = \int_{y_j}^{y_{j+2}} \int_{x_i}^{x_{i+2}} f(x,y)\,dx\,dy$$

we express I as a sum of four double integrals.

$$\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} f(x,y)\,dx\,dy,\quad \int_{y_j}^{y_{j+1}}\int_{x_{i+1}}^{x_{i+2}} f(x,y)\,dx\,dy$$

$$\int_{y_{j+1}}^{y_{j+2}}\int_{x_i}^{x_{i+1}} f(x,y)\,dx\,dy \text{ and } \int_{y_{j+1}}^{y_{j+2}}\int_{x_{i+1}}^{x_{i+2}} f(x,y)\,dx\,dy$$

Applying formula 1 to each of these double integrals and adding the results, we get

$$I = \frac{hk}{4}\Big[f_{i,j} + 2f_{i+1,j} + f_{i+2,j} + 2f_{i,j+1} + 4f_{i+1,j+1} + 2f_{i+2,j+1} + f_{i,j+2} + 2_{i+1,j+2} + f_{i+2,j+2}\Big]$$

The above method can be extended in a natural way when the integral of integration is subdivided into N sub-intervals. This is illustrated in problem ③.

Simpson's one-third rule for double integrals:-

Consider,

$$I = \int_{y_j}^{y_{j+2}}\int_{x_i}^{x_{i+2}} f(x,y)\,dx\,dy$$

Applying simpson's 1/3 rule, we have

$$I = \frac{h}{3}\int_{y_j}^{y_{j+2}} f(x_i,y) + 4f(x_{i+1},y) + f(x_{i+2},y)]\,dy$$

$$= \frac{hk}{9}\Big[f(x_i,y_i) + 4f(x_i,y_{j+1}) + f(x_i,y_{j+2})\Big] +$$

$$4\{f(x_{i+1},y_j) + 4f(x_{i+1},y_{j+1}) + f(x_{i+1},y_{j+2})\}]$$

on simplification, we get

$$I = \frac{hk}{9}\Big[f_{i,j} + f_{i,j+2} + f_{i+2,j} + f_{i+2,j+2} + 4(f_{i,j+1} + f_{i+1,j} + f_{i+1,j+2} + f_{i+2,j+1}) + 16f_{i+1,j+1}\Big]$$

P③ Evaluate $\int_0^1\int_0^1 xy\,dx\,dy$ using (i) Trapezoidal rule (ii) simpson's rule with $h = k = 1/2$

Soln:- Let $f(x,y) = xy$

the values of $f(x,y)$ at the nodal points are given in the following table.

$y \backslash x$	0	0.5	1
0	0	0	0
0.5	0	0.25	0.5
1	0	0.5	1

(i) Trapezoidal rule :

$$I = \frac{hk}{4}\Big[f_{i,j} + 2f_{i+1,j} + f_{i+2,j} + 2f_{i,j+1} + 4f_{i+1,j+1} + 2f_{i+2,j+1} + f_{i,j+2} + 2f_{i+1,j+2} + f_{i+2,j+2}\Big]$$

$$= \frac{(0.5)(0.5)}{4}\Big[4(0.25) + 2(0.5) + 2(0.5) + 1\Big]$$

$$= 0.25$$

(ii) Simpson's rule

$$I = \frac{hk}{9}\Big[f_{i,j} + f_{i,j+2} + f_{i+2,j} + f_{i+2,j+2} + 4(f_{i,j+1} + f_{i+1,j} + f_{i+1,j+2} + f_{i+2,j+1}) + 16 f_{i+1,j+1}\Big]$$

$$= \frac{(0.5)(0.5)}{9}\Big[0+0+0+1+4(0+0+0.5+0.5) + 16(0.25)\Big]$$

$$= 0.25$$

Note :- we observe that in this case both methods gives the exact value for I.

Q. Evaluate $I = \int_0^{1/2}\int_0^{1/2} \frac{\sin xy}{1+xy}\,dx\,dy$ using simpson's rule with $h = k = 1/4$

Soln:- Let $f(x,y) = \frac{\sin xy}{1+xy}$

The values of $f(x,y)$ at the nodal points are given in the following table

$y \backslash x$	0	1/4	1/2
0	0	0	0
1/4	0	0.0588	0.1108
1/2	0	0.1108	0.1979

By Simpson's rule,

$$I = \frac{hk}{9}\Big[f_{i,j}+f_{i,j+2}+f_{i+2,j}+f_{i+2,j+2}+4(f_{i,j+1}+f_{i+1,j}+f_{i+1,j}+f_{i+1,j+2}+f_{i+2,j+1})+16 f_{i+1,j+1}\Big]$$

$$= \frac{1}{144}\big[0+0+0+0.1979+4(0+0+0.1108+0.1108)+16(0.0588)\big]$$

$$= 0.01406$$

Pb Evaluate $I = \int_1^2\int_1^2 \left(\frac{1}{x+y}\right)dx\,dy$ using trapezoidal rule with $h=k=0.25$

Soln:-

The nodal points are given by (x_i, y_j) where $x_i = 1+ih$ and $y_j = 1+jk$ $(i,j = 0,1,2,3,4)$

The values of $f(x,y) = \frac{1}{x+y}$ at the nodal points are given in the following table

y \ x	1	1.25	1.5	1.75	2
1	0.5	0.4444	0.4	0.3636	0.3333
1.25	0.4444	0.4	0.3636	0.3333	0.3077
1.5	0.4	0.3636	0.3333	0.3077	0.2857
1.75	0.3636	0.3333	0.3077	0.2857	0.2667
2	0.3333	0.3077	0.2857	0.2667	0.25

Now,

$I = I_1 + I_2 + I_3 + I_4$ where

$I_1 = \int_1^{1.5}\int_1^{1.5} f(x,y)\,dx\,dy$, $I_2 = \int_1^{1.5}\int_{1.5}^{2} f(x,y)\,dx\,dy$

$I_3 = \int_{1.5}^{2}\int_1^{1.5} f(x,y)\,dx\,dy$ and $I_4 = \int_{1.5}^{2}\int_{1.5}^{2} f(x,y)\,dx\,dy$

By Trapezoidal rule,

$$I_1 = \frac{hk}{4}\Big[f(1,1)+2f(1.25,1)+f(1.5,1)+4f(1.25,1.25)+2f(1.5,1.5)+f(1,1.5)+2f(1.25,1.5)+f(1.5,1.5)\Big]$$

$$= \frac{1}{64}\big[0.5+2(0.4444)+0.4+4(0.4)+2(0.3636)+0.4+2(0.3636)+0.3333\big]$$

$= 0.0871$

By a similar computation we get,

$$I_2 = \frac{1}{64}\Big[f(1.5,1) + 2f(1.75,1) + f(2,1) + 4f(1.75,1.25) + 2f(2,1.25) + f(1.5,1.5) + 2f(1.75,1.5) + f(2,1.5)\Big]$$

$$= \frac{1}{64}\Big[0.4 + 2(0.3636) + 0.3333 + 4(0.3333) + 2(0.3077) + 0.3333 + 2(0.3077) + 0.2857\Big]$$

$= 0.0726$

||ly $I_3 = 0.0726$ and $I_4 = 0.0622$

Hence $I = I_1 + I_2 + I_3 + I_4$

$= 0.0871 + 0.0726 + 0.0726 + 0.0622$

$= 0.2945$

Unit - IV

Numerical solutions of ordinary differential equation

10.1 Taylor's series method

Consider the first order differential equation

$$\frac{dy}{dx} = f(x,y) \quad \text{—— (1)}$$

with $y(x_0) = y_0$

Differentiating (1) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' \qquad \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}\left(\frac{dy}{dx}\right)$$

ie) $y'' = f_x + f_y y'$ —— (2) $= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y'$

Differentiating successively we can obtain $y''', y'''', \ldots$

putting $x = x_0$ and $y = y_0$ we get $y_0', y_0'', y_0''', \ldots$

The Taylor's series expansion of $y(x)$ about $x = x_0$ is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \ldots$$

$$= y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots \quad \text{—— (3)}$$

Substituting the values of $y_0, y_0', y_0'', \ldots$ we obtain $y(x)$ for all values of x for which (3) converges.

Let $x_1 = x_0 + h$ and let

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \ldots$$

once y_1 is known, we can compute $y_1', y_1'', \ldots$

from ① , ② etc

Then y can be expanded in a Taylor's series about $x=x_1$ and we have

$$y(x_1+h) = y(x_2) = y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \quad (4)$$

Continuing in this way ~~are~~ we find the soln $y(x)$

pb using Taylor's method solve $\frac{dy}{dx} = 1+xy$ with $y_0 = 2$

Find (i) $y(0.1)$ (ii) $y(0.2)$ and (iii) $y(0.3)$

Soln:-

(i) The Taylor's algorithm is

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \quad (1)$$

$x_1 = x_0 + h$

Here $x_0 = 0$, $y_0 = 2$ and $h = 0.1$

Given $y' = \frac{dy}{dx} = 1 + xy$ (2)

uv method

$\int uv = u\int v + \int v\,du$

Successively differentiating (2) we get with respect to x

$$y'' = \frac{d^2y}{dx^2} = y + xy'$$

uv

$xy' + y \cdot 1$

$y' + xy'' + y$

$$y''' = 2y' + xy''$$

Now, $y_0' = y'(x_0, y_0) = 1 + x_0 y_0 = 1$

$y_0'' = (y'')(x_0, y_0) = y_0 + x_0 y_0' = 2$ (1)

$y_0''' = (y''')(x_0, y_0) = 2y_0' + x_0 y_0'' = 2$

using these in (1) we get,

$$y_1 = 2 + \frac{(0.1)}{1!} + \frac{(0.1)^2}{2!} \times 2 + \frac{(0.1)^3}{3!} \times 2$$

$y(x) = 2.1103$

$\therefore y(0.1) = 2.1103$

(ii) The Taylor's algorithm for the ~~most~~ next approximation is,

$y_1' = y'(x_1, y_1) = 1 + x_1 y_1 = 1 + (0.1)\,2.1103$
$= 1 \times 2.1103$

$y_1'' = (y'')(x_1, y_1) = y_1 + x_1 y_1' = 2.1103 + (0.1)(1.21103)$
$= 2.2314$

$y_1''' = y'''(x_1, y_1) = 2y_1' + x_1 y_1'' = 2(1.21103) + (0.1)(2.2314)$
$= 2.6452$

$\therefore$ (3) becomes

$$y_2 = 2.1103 + \frac{(0.1)}{1!}(1.21103) + \frac{(0.1)^2}{2!}(2.2314) + \frac{(0.1)^3}{3!}(2.6452)$$

$= 2.2430$

$\therefore y(0.2) = 2.2430$

(iii) The Taylor's algorithm for third approximation is,

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' \quad \text{—— (4)}$$

Now,

$$y_2' = y'(x_2, y_2) = 1 + x_2 y_2 = 1.4486$$

$$y_2'' = y''(x_2, y_2) = y_2 + x_2 y_2' = 2.53272$$

$$y_2''' = y'''(x_2, y_2) = 2y_2' + x_2 y_2'' = 3.4037$$

(4) becomes,

$$y_3 = 2.2430 + (0.1)(1.4486) + \frac{(0.1)^2}{2!}(2.53272) + \frac{(0.1)^3}{3!}(3.4037)$$

$$= 2.4011$$

$$\therefore y(0.3) = 2.4011$$

P6 Using Taylor's method, find $y(0.1)$ correct to 3 decimal places from $\frac{dy}{dx} + 2xy = 1$, $y_0 = 0$

Soln:-

Given $\frac{dy}{dx} = y' = 1 - 2xy$ —— (1)

The Taylor's algorithm is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Here $x_0 = 0$, $y_0 = 0$ and $h = 0.1$. Now successively differentiating (1), we get

$$y'' = -2(y + xy')$$

$$y''' = -2[xy'' + 2y']$$

$$\therefore y'(x_0, y_0) = y_0' = 1 \Rightarrow 1 - 2xy = 1 - 0 = 1$$

$$y''(x_0, y_0) = y_0'' = 0 \Rightarrow -2(y_0 + x_0 y_0') = -2(0+0) = 0$$

$$y'''(x_0, y_0) = y_0''' = -4 \Rightarrow -2[x_0 y_0'' + 2y_0'] = -2[0(0) + 2(1)] = -4$$

Substituting the value $y_0', y_0'', \ldots$ we get

$$y_1 = 0 + 0.1 + \frac{(0.1)^2}{2!} \times 0 + \frac{(0.1)}{3!}(-4)$$

$$= 0.0993$$

$$\therefore y(0.1) = 0.0993$$

P6 Using Taylor's series method find y at $x = 1.1$ and 1.2 by solving $\frac{dy}{dx} = x^2 + y^2$ given $y(1) = 2.3$

Soln:-

Given $\frac{dy}{dx} = x^2 y^2$ —— (1)

Here $x_0 = 1$, $y_0 = 2.3$ and $h = 0.1$

(i) The Taylor's Series expansion is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad \text{——} (2)$$

Differentiating (1) successively with respec to x we get,

$$y'' = 2x + 2yy'$$

$$y''' = 2 + 2(yy'' + y'^2)$$

$$y_0' = x_0^2 + y_0^2 = 6.29$$

$$y_0'' = 2x_0 + 2y_0 y_0' = 30.934$$

$$y_0''' = 2 + 2(y_0 y_0'' + y_0'^2) = 223.4246$$

using this in (2) we get,

$$y(1.1) = y_1 = 2.3 + \frac{0.1}{1}(6.29) + \frac{(0.1)^2}{2}(30.934) + \frac{(0.1)^3}{6}(223.4246)$$

$$= 3.1209$$

Here $x_1 = 1.1$ and $y_1 = 3.1209$

(ii) we have the Taylor's series expansion

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \quad \text{——} (3)$$

$$y_1' = y'(x_1, y_1) = x_1^2 + y_1^2 = 10.95$$

$$y_1'' = y''(x_1, y_1) = 2x_1 + 2y_1 y' = 70.5477$$

$$y_1'' = y''(x_1, y_1) = 2x_1 + 2y_1 y' = 70.5477$$

$$y_1''' = (y''')(x_1, y_1) = 2 + 2(y_1 y_1'' + y_1'^2)$$

$$= 682.1496$$

$\therefore$ (3) becomes

$$y_2 = 3.1209 + 0.1(10.95) + \frac{(0.1)^2}{2}(70.5477) + \frac{(0.1)^3}{6}(682.1496)$$

$$= 4.6823$$

Hence

$y(1.1) = 3.1209$ and $y(1.2) = 4.6823$

10.2 picard's method

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{——} (1)$$

with initial condition $y = y_0$ when $x = x_0$

we now replace (1) by an equivalent integral equation.

Integration (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

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ie) $y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$ ——— (2)

This is an integral equation which contains the unknown y under the integral sign.

(2) is equivalent to (1) since any soln of (2) is a soln of (1) and vice verse.

The first approximation y_1 to the soln is obtained by putting $y=y_0$ in $f(x,y)$ and from (2) we have

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx$$

|||ly for the second approximation y_2, put $y=y_1$ in $f(x,y)$ and from (2) we have

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx$$

continuing this process the n^{th} approximation is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x,y_{n-1})\,dx$$

This is known as picard's interation formula.

Note:-

picard's method gives a sequence of approximations $y_1, y_2, \ldots$ each giving a better result than the preceeding one. But this can be applied only to equations in which the successive integration can be obtained easily.

Pb using picard's method solve $\frac{dy}{dx} = 1+xy$ with $y(0)=2$. Find $y(0.1)$, $y(0.2)$ and $y(0.3)$

Text

soln:- The picard's interaction formula for the differential equation.

$\frac{dy}{dx} = f(x,y)$ is $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$ where $n=1,2\ldots$

Given $f(x,y) = 1+xy$, $x_0=0$ and $y_0=2$

$\therefore$ The first approximation is (1) ———

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx$$

$\Rightarrow f(x,y_0) = 1+xy_0 = 1+x\cdot 2$

$$= 2 + \int_0^x (1+2x)\,dx$$

$$= 2+x+x^2$$

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The second approximation is,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 2 + \int_{0}^{x} [1 + x(2 + x + x^2)]\,dx$$

$$= 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4}$$

The third approximation is,

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= 2 + \int_{x_0}^{x} [1 + x(2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4})\,dx$$

$$= 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{26} \quad \text{--- (1)}$$

putting $x = 0.1, 0.2$ and 0.3, in (1) we get

$y_1 = y(0.1) = 2.1104$

$y_2 = y(0.2) = 2.2431$

$y_3 = y(0.3) = 2.4012$

Pb find the value of $y(0.1)$ by picard's method given

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad y(0) = 1$$

Soln:- The picard's interative formulla for the differential equation $\frac{dy}{dx} = f(x, y)$ is

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad \text{where } n = 1, 2, 3 \ldots$$

Here $f(x, y) = \frac{y - x}{y + x}$, $x_0 = 0$ and $y_0 = 1$

∴ The first approximation is

$$y_1 = y_0 + \int_{0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_{0}^{x} \left(\frac{1 - x}{1 + x}\right) dx$$

$$= 1 + \int_{0}^{x} \left(-1 + \frac{2}{1 + x}\right) dx \quad (\text{By partial fraction})$$

$$= 1 + [-x + 2\log_e(1 + x)]_0^x$$

$$= 1 - x + 2\log_e(1 + x)$$

Putting $x = 0.1$ we get $y_1 = y(0.1)$

$$= 1 - 0.1 + 2\log_e(1 + 0.1)$$

$= 0.9 + 2 \times 0.0953$

$= 1.0906$.

Pb Find the successive approximate soln of the differential equation $y' = y$, $y(0) = 1$ by picard's method and compare it with the exact soln.

Soln:- picard's iteration formula is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad \text{where } n = 1, 2 \ldots$$

Here $f(x,y) = y$, $x_0 = 0$ and $y_0 = 1$

∴ The first approximation soln is

$$y_1 = y_0 + \int_0^x f(x, y_0)\,dx$$

$$= 1 + \int_0^x dx = 1 + x$$

The second approximate soln is

$$y_2 = y_0 + \int_0^x y_1\,dx$$

$$= 1 + \int_0^x (1+x)\,dx$$

$$= 1 + x + \frac{x^2}{2}$$

The third approximate soln is

$$y_3 = y_0 + \int_0^x y_2\,dx$$

$$= 1 + \int_0^x (1+x)\,dx = 1 + x + \frac{x^2}{2}$$

The third approximate soln is,

$$y_3 = y_0 + \int_0^x y_2\,dx$$

$$= 1 + \int_0^x \left(1 + x + \frac{x^2}{2}\right)dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

proceeding like this we can find the successive approximations.

Given differential equation is $y' = y$

(i) $\frac{dy}{dx} = y$

∴ $\frac{dy}{y} = dx$

Integrating we get,

$$\log_e y = x + c,$$

∴ The exact soln is $y = e^{x + c_1} = ce^{x}$

using the initial condition $x=0, y=1$ we get $c=1$

ie) $y = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

Hence the Successive approximative solutions are the partial sums of the exact solution.

Pb find an approximate solution of the initial value problem $y' = 1+y^2, y(0)=0$. by picard's method and compare with the exact solution.

soln:- picard's iteration formula is given by.

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad \text{where } n=1,2\ldots$$

Here $f(x,y) = 1+y^2, x_0=0$ and $y_0=0$

$\therefore$ The first approximation is $y_1 = y_0 + \int_0^x f(x,y)\,dx$

$$= y_0 + \int_0^x (1+y_0^2)\,dx$$

$$= \int_0^x dx = x.$$

The second approximation is,

$$y_2 = y_0 + \int_0^x (1+y_1^2)\,dx$$

$$= \int_0^x (1+x^2)\,dx = x+\frac{x^3}{3}$$

The third approximation is,

$$y_3 = y_0 + \int_0^x (1+y_2^2)\,dx$$

$$= \int_0^x \left[1+\left(x+\frac{x^3}{3}\right)^2\right]dx = \int_0^x \left[1+x^2+\frac{2x^4}{3}+\frac{x^6}{9}\right]dx$$

$$= x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{x^7}{63}$$

proceeding like this we can find the further approximate solution.

Now, the given differential equation is $\frac{dy}{dx} = 1+y^2$

ie) $\frac{dy}{1+y^2} = dx$

Integrating we get, $\tan^{-1} y = x+c$

using the initial condition we get $c=0$

$\therefore y = \tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots$

clearly the first three terms of y_3 are some as that of the exact solution.

10.3 Euler's method

Taylor's series method and Picard's method that we have discussed in the previous two sections yield the soln of a differential equation in the form of a power series. we now proceed to describe methods which give the soln in the form of table values at equally spaced points.

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0 \quad \text{——} (1)$$

Suppose we want to solve (1) for y; at the points $x_r = x_0 + rh$, $r = 1, 2, 3 \ldots$

Integrating (1) between the limits x_0 and x_1 we get,

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

Hence, $y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\,dx$ —— (2)

Assuming that $f(x,y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$, we get,

$$y_1 = y_0 + (x_1 - x_0) f(x_0, y_0)$$

$$\therefore y_1 = y_0 + hf(x_0, y_0) \quad \text{——} (3)$$

III'ly if $x_1 \le x \le x_2$, we have

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x,y)\,dx$$

Substituting $f(x_1, y_1)$ for $f(x,y)$ we get

$$y_2 = y_1 + hf(x_1, y_1) \quad \text{——} (4)$$

proceeding like this we obtain the general formula

$$y_{n+1} = y_n + h f(x_n, y_n) \quad \ldots \quad n = 0, 1, 2 \ldots$$

This is called Euler's algorithm since $x_n = x_0 + nh$ and $y_n = y(x_n)$, the above formula can be also be written as

$$y(x+h) = y(x) + hf(x,y)$$

Modified Euler's method

Instead of approximating $f(x,y)$ by $f(x_0, y_0)$ in (2) we approximate it by $\frac{1}{2}[f(x_0, y_0) + f(x_1, y_1)]$ which is the mean of the slopes of the tangents at the points corresponding to $x = x_0$, and $x = x_1$. Thus we obtain

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$ which is the mean of the slopes of the tangents at the points corresponding to $x = x_0$, and $x = x_1$. Thus we obtain

~~$y_1^{(1)}$~~ where y_1 is given by ② $y_1^{(1)}$ is the first modified value of y_1.

Let $y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

we repeat this process till two consecuite values of y agree. Let y_1 be the final value obtained to the desired acuracy. using this value of y_1 we compute.

$$y_2 = y_1 + h f(x_0 + h, y_1)$$

Now, let $y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_1)]$

we repeat this process until two consecuite values agree. Then we proceed to calculate y_3, as above and continue the process till we calculate y_n.

Pb Solve $\frac{dy}{dx} = 1 - y$, $y(0) = 0$ using Euler's method. find y at $x = 0.1$ and $x = 0.2$. Compare the result with results of the exact solution.

Assignment

Soln: The Euler's formula for the numerical soln of the differential equation $\frac{dy}{dx} = f(x, y)$ is

$$y_{n+1} = y_n + h f(x_n, y_n) \longrightarrow ①$$

The given differential equation is $\frac{dy}{dx} = 1 - y$.

$x_1 = 0.1$

$x_2 = 0.2$

$h = x - x_0$

$= 0.1 - 0$

$\boxed{h = 0.1}$

$\therefore f(x, y) = 1 - y$, $h = 0.1$

Also we have $x_0 = 0$, $y_0 = 0$, $h = 0.1$

putting $n = 0$ in ① we get $y(0.1) = y_1$

$= y_n + h f(x_n, y_n)$

$= y_0 + h f(x_0, y_0)$

$= 0 + 0.1(1) = 0.1$

Now,

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

putting $n = 1$ in ① we get $y(0.2) = y_2$

$= y_1 + f(x_1, y_1)$

$= 0.1 + (0.1)(1 - 0.1)$

$= 0.19$

Hence $y(0.1) = 0.1$ and $y(0.2) = 0.19$

The exact solution of $\frac{dy}{dx} = 1 - y$ is not from $\frac{dy}{1 - y} = dx$

$\therefore \log(1 - y) = x + c$

putting $x=0$ and $y=0$ we get $c=0$

$\therefore 1-y=e^{x}$. Hence $y=1-e^{x}$

$y(0.1)=1-e^{0.1}=0.1052$ and $y(0.2)=1-e^{0.2}$
$=0.2214$

Q) Using Euler's method solve $\frac{dy}{dx}=1+xy$ with $y(0)=2$. Find $y(0.1)$, $y(0.2)$ and $y(0.3)$. Also find the values by modified Euler's method.

Soln:- The Euler's formula for numerical solution of the differential equation

$\frac{dy}{dx}=f(x,y)$ is

$$y_{n+1}=y_n+hf(x_n,y_n) \quad n=0,1,2\dots \quad \text{——— (1)}$$

Here $f(x,y)=1+xy$, $x_0=0$, $y_0=2$ and $h=0.1$

putting $n=0$ in (1) we get $y(0.1)=y_1$

$=y_0+hf(x_0,y_0)$

$=2+(0.1)f(0,2)$

Now $x_1=x_0+h=0.1$ $\qquad = 2.1$

putting $n=1$ in (1) we get $y(0.2)=y_2$

$=y_1+hf(x_1,y_1)$

$=2.1+(0.1)[1+0.1\times 2.1]$

Now, $\qquad =2.221$

$x_2=x_1+h=0.2$

putting $n=2$ in (1) we get $y(0.3)=y_3$

$=y_2+hf(x_2,y_2)$

$=\cancel{2.1+(0.1)[1+0.1\times 2.1]}$

$=2.221+(0.1)[1+0.2\times 2.221]$

$=2.3654$

Hence $y(0.1)=2.1$, $y(0.2)=2.2221$ and

$y(0.3)=2.3654$

Modified Euler's method

Starting value for $y_1=2.1$

$y_1^{(1)}=y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1)]$

$=2+\frac{0.1}{2}[1+1+(0.1)(2.1)]$

$=2.2205$

$y_1^{(2)}=y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(1)})]$

$$= 2 + \frac{0.1}{2}\left[1 + 1 + (0.1)(2.2205)\right]$$

$$= 2.11111$$

Continuing this process, we get $y_1^{(3)} = 2.1105$,

$y_1^{(4)} = 2.1105$

$\therefore$ Final value of $y_1 = 2.1105$

Now, Starting value of

$$y_2 = y_1 + h\, f(x_0 + h, y_1)$$

$$= 2.1105 + (0.1)\left[1 + (0.1)(2.1105)\right]$$

$$= 2.2316$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_0+h, y_1) + f(x_0 + 2h ; y_2)\right]$$

$$= 2.1105 + \frac{0.1}{2}\left[1 + (0.1)(2.1105) + 1 + (0.2)(2.2316)\right]$$

$$= 2.2434$$

Continuing this process we get, $y_2^{(2)} = 2.2435$, $y_2^{(3)} = 2.2434$

$\therefore$ Final value of $y_2 = 2.2434$

Starting value of

$$y_3 = y_2 + h\, f(x_0 + 2h, y_2)$$

$$= 2.2434 + (0.1)\left[1 + (0.2)(2.2434)\right]$$

$$= 2.2579$$

$$y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_0+2h, y_2) + f(x_0 + 3h, y_3)\right]$$

$$= 2.2434 + \frac{0.1}{2}\left[1 + (0.2)(2.2434) + 1 + (0.3)(2.2579)\right]$$

$$= 2.3997$$

Continuing this process we get $y_3^{(2)} = 2.4018$, $y_3^{(3)} = 2.4019$,

$y_3^{(4)} = 2.4019$.

$\therefore$ final value of $y_2 = 0.9857$

Starting value of $y_3 = y_2 + 0.1\left[f(1.2, 0.9857)\right]$

$$= 0.9730$$

Now, $y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_0 + 2h, y_2) + f(x_0 + 3h, y_3)\right]$

$=$

$\therefore$ Final value of $y_3 = 2.4019$

Hence $y_1 = 2.1105$, $y_2 = 2.3997$ and $y_3 = 2.4019$.

Given $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, $y(1) = 1$. Evaluate $y(1.3)$ by modified Euler's method.

Soln:- $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x} = \frac{1 - xy}{x^2}$, $y(1) = 1$

$\therefore f(x,y) = \frac{1-xy}{x^2}$, $x_0 = 1, y_0 = 1$ and we take $h = 0.1$,

Starting value of $y(1.1) = y_1$ is given by.

$$y_1 = y_0 + h f(x_0, y_0) = 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right]$$

$$= 1 + \frac{0.1}{2}\left[0 + f(1.1)\right]$$

$$= 1 + 0.05\left[\frac{1-1.1}{(1.1)^2}\right] = 0.9959$$

$$y_1^{(2)} = 1 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.05\left[0 + \frac{1-(1.1)\times(0.9959)}{(1.1)^2}\right]$$

$$= 0.9605$$

Continuing this process, we obtain $y_1^{(3)} = 0.9977$

$y_1^{(4)} = 0.9960$, $y_1^{(5)} = 0.9960$

$\therefore$ Final value of $y_1 = 0.9960$

Now,

Starting value of $y_2 = y_1 + h f(x_0 + h, y_1)$

$$= 0.9960 + 0.1\left[\frac{1-(1.1)(0.9960)}{(1.1)^2}\right]$$

Now, $= 0.9881$

$$y_2^{(1)} = y_1 + h/2\left[f(x_0+h, y_1) + f(x_0+2h, y_2)\right]$$

$$= 0.9960 + (0.05)\left[f(1.1, 0.9960) + f(1.2, 0.9881)\right]$$

$$= 0.9856$$

Continuing this process we obtain $y_2^{(2)} = 0.9857$,

$y_2^{(3)} = 0.9857$

$\therefore$ Final value of $y_2 = 0.9857$

Starting value of $y_3 = y_2 + 0.1\left[f(1.2, 0.9857)\right]$

$$= 0.9730$$

Now,

$$y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_0+2h, y_2) + f(x_0+3h, y_3)\right]$$

$$= 0.9857 + 0.05\left[f(1.2, 0.9857) + f(1.3, 0.9730)\right]$$

$$= 0.971$$

Continuing this process, we get $y_3^{(2)} = 0.9716$

$y_3^{(3)} = 0.9662$, $y_3^{(4)} = 0.9662$

$\therefore y(1.3) =$ final value of $y_3 = 0.9662$.

10.4 Runge-kutta methods

First order R-K method

consider $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ ——— ①

The Euler's formula for first approximation to the solution of the above differential equation is given by.

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h y_0' \quad [\because y' = f(x, y)]$$

Also $y_1 = y(x_0+h) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots$

Clearly the Euler's method agrees with the Taylor's series solution upto the term in h. Hence Euler's method is the Runge-kutta method of first order.

II second order R-K method

The modified Euler's formula for ① is

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_0+h f(x_0, y_0))\right]$$

i.e) $y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0+h, y_0+h f_0)\right]$

where $f_0 = f(x_0, y_0)$

expanding the L.H.S by Taylor's series we get

$$y_1 = y(x_0+h) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \text{ ③}$$

expanding $f(x_0+h, y_0+h f_0)$ by Taylor's series for a function of two variables we have

$$f(x_0+h, y_0+h f_0) = f(x_0, y_0) + \frac{h}{1!}\left[\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + f_0\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)}\right] + o(h^2)$$

using this in ② we get,

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + h f_0\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} + o(h^2)\right]$$

$$= y_0 + \frac{1}{2}\left[h f_0 + h f_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + f_0\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)}\right\} + o(h^3)\right]$$

$$= y_0 + h f_0 + \frac{h^2}{2!} f_0' + o(h^3)$$

$$= y_0 + h y_0' + \frac{h^2}{2!} y_0'' + o(h^3)$$

Comparing ③ and ④ we find that the modified Euler's method agrees with the Taylor's series solution upto the h^2 term.

Hence the modified Euler's method is the Runge-kutta method of second order.

$\therefore$ The Second order Runga-kutta formula is

$$\boxed{y_1 = y_0 + \frac{1}{2}(k_1 + k_2)}$$

where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0+h, y_0+k_1)$

The third order Runga-kutta formula is give by

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)}$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + h, y_0 + k')$

where $k' = hf(x_0+h, y_0+k_1)$

Fourth order R-K method :-

This method is most commonly used and is referred as the Runge-kutta method

The working rule for solving the initial value problem

$\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

by 4th order Runge-Kutta method is follows :

Calculate successively.

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

and $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Then the required approximate value is given by

$$\boxed{y_1 = y_0 + \Delta y}$$

$|||^{ly}$ the value of y in the second interval is obtained by replacing x_0 by x_1 and y_0 by y_1 in the above set of formulae and we obtain y_2.

In general to find y_n substitute x_{n-1}, y_{n-1} is the expression for k_1, k_2 etc.

Note:- ① The operation is identical whether the differential

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equation is linear (or) non-linear.

Note :- 2

To evaluate y_{n+1} we need information only at the point y_n. Information at the points y_{n-1}, y_{n-2} etc. are not directly required. Hence R.K methods are step methods.

Pb Compute $y(0.1)$ and $y(0.2)$ ~~and~~ by Runge-kutta method by 4th order for differential equation.

$$\frac{dy}{dx} = xy + y^2 , \quad y(0)=1$$

$y(x_0) = y_0$, $x_0 = 0$, $y_0 = 1$

Soln:- The formula for the fourth order Runge-kutta method are

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$k_4 = hf(x_0+h, y_0+k_3)$$

and $\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

where h is the interval of ~~differentiating~~ differencing and (x_0, y_0) is the initial value. Here $f(x,y) = xy+y^2$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

$(xy+y^2) = ((0)(1)+(1)^2) = 0+1 = 1 = (0.1)(1) = 0.1$

Now, $k_1 = (0.1)(0+1) = 0.1$

$k_2 = (0.1)[0.05(1.05)+(1.05)^2] = 0.1155$

$k_3 = (0.1)[0.05(1.05775)+(1.05775)^2] = 0.1172$

$k_4 = (0.1)[(0.1)(1.1172)+(1.1172)^2]$

$k_4 = 0.1360$

$\Delta y = \frac{1}{6}[0.1+0.2310+0.2344+0.1360] = \frac{1}{6}(0.7014)$

$\Delta y = 0.1169$ $\quad y_0 = y_1 - \Delta y_0$

$\therefore y_1 = y_0 + \Delta y_0 = 1+0.1169 = 1.1169$

$\therefore y(0.1) = 1.1169$ $\quad y_1 = 1.1169$ $\quad x_1 = 0.1$

For the second approximation, we have $y(x_1)$ y_1

$k_1 = hf(x_1, y_1) = 0.1[0.1\times(1.1169)+(1.1169)^2]$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= (0.1)[0.15(1.1849) + (1.1849)^2]$

$k_2 = 0.1582$

$k_3 = hf(x_1 + \frac{h}{2}, \frac{k}{2} + y_1)$

$= (0.1)[0.15(1.196) + (1.196)^2]$

$k_3 = 0.1610$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)[0.2(1.2779) + (1.2779)^2]$

$k_4 = 0.1889$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1359 + 0.3164 + 0.3220 + 0.1889)$

$\Delta y = 0.1605$

$\therefore \; y_2 = y_1 + \Delta y = 1.1169 + 0.1605 = 1.2774$

$y(0.2) = 1.2774$ //

Pb Use Runge-kutta method of the fourth order to find $y(0.1)$ given that $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0) = 1$

Soln:-

The formula for the fourth order Runge-kutta method is given by

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h/2, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$\Delta y = 1/6(k_1 + 2k_2 + 2k_3 + k_4)$

where h is the interval of diffencing and (x_0, y_0) is the initial value.

Here $f(x, y) = \frac{1}{x+y}$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

Now, $k_1 = (0.1)\left(\frac{1}{0+1}\right) = 0.1$ $\quad k_1 = hf(x_0, y_0)$

$= (0.1) f(0, 1) = (0.1)\left(\frac{1}{0+1}\right) = 0.1$

$k_2 = (0.1)\left[\frac{1}{(x_0 + \frac{h}{2}) + (y_0 + \frac{k_1}{2})}\right]$

$$= \frac{0.1}{0.5+1.05} = (0.1)\left[\frac{1}{1.1}\right] = 0.0909$$

$$k_3 = (0.1)\left[\frac{1}{(x_0+\frac{h}{2})+(y_0+\frac{k_2}{2})}\right]$$

$$= \frac{0.1}{0.05+1.045} = 0.0913$$

$$k_2 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{k_1}{2})$$
$$= (0.1)f\left(0+\tfrac{0.1}{2}, 1+\tfrac{0.1}{2}\right)$$
$$= (0.1)f(0.05, 1.05)$$
$$= (0.1)\left(\frac{1}{0.05+1.05}\right)$$
$$= 0.0909$$

$$k_4 = (0.1)\left[\frac{1}{(x_0+h)+(y_0+k_3)}\right]$$

$$= \frac{(0.1)}{0.1+1.0913} = 0.0839$$

$$\therefore \Delta y = \frac{1}{6}\left[0.1+2(0.0909)+2(0.0913)+0.0839\right]$$

$$= 0.0914$$

$$\therefore y_1 = y_0+\Delta y = 1+0.0914 = 1.0914$$

$$\therefore y(0.1) = 1.0914$$

15) given $y' = x^2-y$, $y(0)=1$ find $y=(0.1)$ using Runge kutta fourth order.

$y(x_0)=y_0$

$x_0=0$, $y_0=1$

proof:-

The formula for the fourth order Runge-kutta method is given by

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$k_4 = hf(x_0+h, y_0+k_3) \text{ and}$$

$$\Delta y = \tfrac{1}{6}(k_1+2k_2+2k_3+k_4)$$

where h is the interval of differencing and (x_0, y_0) is the initial value.

Here $f(x,y) = x^2-y$, $x_0=0$, $y_0=1$, $h=0.1$

$$\therefore k_1 = (0.1)(-1) = -0.1$$

$$k_2 = (0.1)\left[\left(\frac{0.1}{2}\right)^2 - \left(1+\frac{(-0.1)}{2}\right)\right]$$

$$= 0.1[0.0025-0.95] = -0.0948$$

$$k_3 = (0.1)\left[\frac{(0.1)^2}{2} - \left(1+\frac{(-0.0948)}{2}\right)\right]$$

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$= -0.095$

$k_4 = (0.1)[(0.1)^2 - (1-0.095)]$

$= -0.0895$

$\therefore \Delta y = \frac{1}{6}[-0.1 - 0.1896 - 0.190 - 0.0895]$

$= -0.09485$

$\therefore y(0.1) = y_0 + \Delta y = 1 - 0.09485$

$= 0.9052$

Pb using Runge-kutta method of fourth order for $y(0.1)$, $y(0.2)$ and $y(0.3)$ given that $\frac{dy}{dx} = 1 + xy$, $y(0)=2$

Soln:-

The formula for the 4^{th} order Runge-kutta method of the differential equation $\frac{dy}{dx} = f(x,y)$ are

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where h is the interval of differencing and (x_0, y_0) is the initial value

Here $f(x,y) = 1 + xy$, $x_0 = 0$, $y_0 = 2$ and $h = 0.1$

$\therefore k_1 = (0.1)(1+0) = 0.1$

$k_2 = (0.1)\left[1 + \left(0 + \frac{(0.1)}{2}\right)\left(2 + \frac{0.1}{2}\right)\right]$

$= 0.1[1 + 0.1025]$

$= 0.11025$

$k_3 = 0.1\left[1 + \left(0 + \frac{0.1}{2}\right)\left(2 + \frac{0.11025}{2}\right)\right]$

$= 0.1103$

$k_4 = (0.1)\left[1 + \left(0 + \frac{0.1}{2}\right)(2 + 0.1103)\right]$

$= 0.1106$

$\Delta y = \frac{1}{6}[0.1 + 2(0.11025) + 2(0.1103) + 0.1106]$

$= 0.1086$

$\therefore y_1 = y_0 + \Delta y$

$= 2.1086$

$\therefore y(0.1) = 2.1086$

$y(x_1) = y_1$

For the second approximation we have,

$k_1 = hf(x_1, y_1) \quad x_1 = 0.1, \ y_1 = 2.1086$

$= (0.1)[1 + (0.1)(2.1086)] = 0.1211$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= (0.1)\left[1 + (0.1 + \frac{0.1}{2})\left(2.1086 + \frac{0.1211}{6}\right)\right]$

$= 0.1325$

$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right]$

$= (0.1)\left[1 + (0.1 + \frac{0.1}{2})(2.1086 + \frac{0.1325}{2})\right]$

$= \cancel{0.1464} \ 0.1326$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= (0.1)[1 + (0.1 + 0.1)(2.1086 + 0.1326)]$

$= 0.1464$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1211 + 2(0.1325) + 2(0.1326) + 0.1464]$

$= 0.1330$

$\therefore y_2 = y_1 + \Delta y = 2.2416$

For the third approximation we have

$k_1 = hf(x_2, y_2)$

$= (0.1)[1 + (0.2)(2.2416)] = 0.1448$

$k_2 = hf(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2})$

$= (0.1)\left[1 + (0.2) + (\frac{0.1}{2})\right]\left[2.2416 + \frac{0.1448}{2}\right].$

$k_2 = 0.1579$

$k_3 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right]$

$= (0.1)\left[1 + (0.2 + \frac{0.1}{2})\left(2.2416 + \frac{0.1579}{2}\right)\right.$

$= 0.158$

$k_4 = hf(x_2+h, y_2+k_3)$

$= (0.1)[1+(0.2+0.1)(2.2416+0.158)]$

$= 0.158$

$\therefore \Delta y = 1/6($

$k_4 = hf(x_2+h, y_2+k_3)$

$= (0.1)[1+(0.2+0.1)(2.2416+0.158)]$

$= 0.172$

$\Delta y = 1/6(k_1+2k_2+2k_3+k_4)$

$= \frac{1}{6}(0.1448+2(0.1579)+2(0.158)+0.172]$

$= 0.1581$

$\therefore y_3 = y_2+\Delta y = 2.3997$

Hence we have $y(0.1) = 2.1086$, $y(0.2) = 2.2416$ and $y(0.3) = 2.3397$

Pb using 4th order Runge-Kutta method, evaluate the value of y. when $x = 1.1$ given that

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \; y(1) = 1$$

Soln:-

The formula for the 4th order Runge-Kutta method of the differential equation

$\frac{dy}{dx} = f(x,y)$ is given by

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$

$k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$

$k_4 = hf(x_0+h, y_0+k_3)$

$\Delta y = 1/6(k_1+2k_2+2k_3+k_4)$

where h is the interval of differencing and (x_0, y_0) is the initial value.

Here $f(x,y) = 1/x^2 - \frac{y}{x}$, $x_0 = 1$ and $y_0 = 1$, $h = 0.1$

Now, $k_1 = (0.1)(\frac{1}{1^2} - \frac{1}{1}) = 0$

$k_2 = (0.1)\left(\frac{1}{(x_0+h/2)^2} - \frac{y_0+k_1/2}{x_0+h/2}\right)$

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$$=(0.1)\left(\frac{1}{(1+\frac{0.1}{2})^2} - \frac{1+0}{1+\frac{0.1}{2}}\right)$$

$$=(0.1)(0.9070-0.9524)$$

$$=-0.00454$$

$$k_3 = (0.1)\left(0.9070 - \frac{1+\left(\frac{-0.00454}{2}\right)}{1.05}\right)$$

$$=(0.1)(0.9070-0.9502)$$

$$=-0.00432$$

$$k_4 = (0.1)\left(\frac{1}{(1.1)^2} - \frac{1-0.00432}{1.1}\right)$$

$$=(0.1)(0.8264-0.9052)$$

$$=-0.00788$$

$$\therefore \Delta y = 1/6\,(0-0.00908-0.00864-0.00788)$$

$$=-0.0042667$$

$$\therefore y_1 = y(1.1) = y_0+\Delta y = 1+(-0.0042667)$$

$$\therefore y = 0.9957$$

Unit-V

10.5 predictor Correct methods :-

Consider the equation $\frac{dy}{dx} = f(x,y)$ with $y(x_0)=y_0$. we divide the range for x into a number of step sizes of equal ~~witht~~ width h. If x_i and x_{i+1} are two consecutive points then $x_{i+1} = x_i+h$.

Euler's formula for the above differential equation is

$$y_{i+1} = y_i + hf(x_i, y_i) \quad i=1,2,3\ldots \quad -(1)$$

The modified Euler's formula is,

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right] \quad i=1,2\ldots \quad -(2)$$

Equation (1) is called the predictor and (2) is called corrector.

A predictor formula is used to predict the value y_{i+1} of y_{i+1} of y at x_{i+1} and then corrector formula is used to improve the value of y_{i+1}

10.6 Milne's method

Consider the first order differential equation

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0.$$

Newton's forward difference formula canbe written as

$$f(x,y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots \quad (1)$$

Substituting this in the relation

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x,y)\,dx$$

we get

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right] dx$$

Put $x = x_0 + nh$, differentiating Hence $dx = h\,dn$

when $x = x_0$, $n = 0$ and when $x = x_0 + 4h$, $n = 4$

$$\therefore y_4 = y_0 + h\int_0^4 \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right]$$

$nf_0 + \frac{n^2}{2}\Delta f_0 + \frac{n^2 - n}{2}\Delta^2 f_0$ integrate $\frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 f_0$

$$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \cdots\right]$$

$$= y_0 + h\left[4y_0' + 8(E-1)y_0' + \frac{20}{3}(E^2 - 2E - 1)y_0' + \frac{8}{3}(E^3 - 3E^2 + 3E - 1)y_0'\right]$$

(neglacting fourth and higher order difference)

$$= y_0 + h\left[4y_0' + 8(y_1' - y_0') + \frac{20}{3}(y_2' - 2y_1' + y_0') + \frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0')\right]$$

$$= y_0 + h\left[\frac{8}{3}y_1' - \frac{4}{3}y_2' + \frac{8}{3}y_3'\right]$$

$$= y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

Thus $y_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$

Since $x_0, x_1, \ldots x_4$ are any five consecutive values of x the above equations can be written as

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}' - y_{n-1}' + 2y_n'\right]$$

This is called Milne's predictor formula.

(the subscript p indicates that it is a predicate)

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value)

This formula can be used to predicate the value of y_4 when those of y_0, y_1, y_2, y_3 are Known.

To get a corrector formula we substitude Newton's formula ① in the relation.

$$y_2 = y_0 + \int_{x_0}^{x_0+2h} f(x,y)\,dx$$

and we get,

$$y_2 = y_0 + \int_{x_0}^{x_0+2h} \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right] dx$$

$$= y_0 + h\int_0^2 \left[f_0 + h\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right]$$

putting $x = x_0 + nh$

$$= y_0 + h\left[2f_0 + 2\Delta f_0 + \tfrac{1}{3}\Delta^2 f_0\right]$$

$$= y_0 + h\left[2y_0' + 2(E-1)y_0' + \frac{1}{3}(E^2 - 2E + 1\ y_0')\right]$$

neglacting higher order differences.

$$= y_0 + h\left[2y_0' + 2(y_1' - y_0') + \frac{1}{3}(y_2' - 2y_1' + y_0')\right]$$

Thus,

$$y_2 = y_0 + \frac{h}{3}\left[y_0' + 4y_1' + y_2'\right]$$

Since x_0, x_1, x_2 are any five consecutive values of x the above equations can be written as,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[y_{n-2}' - y_{n-1}' + 2y_n'\right]$$

This is called Milne's predictor formula.

(The subscript p indicates that it is a predicated value)

This formula can be used to predicate the value of y_4 when those of y_0, y_1, y_2, y_3 are known.

To get a corrector formula we substitude Newton's formula ① in the relation.

$$y_2 = y_0 + \int_{x_0}^{x_0+2h} f(x,y)\,dx$$

and we get,

$$y_2 = y_0 + \int_{x_0}^{x_0+2h} \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right] dx$$

$$= y_0 + h \int_0^2 \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!} \Delta^2 f_0 + \cdots \right] \quad (\text{putting } x = x_0 + nh)$$

$$= y_0 + h \left[2f_0 + 2\Delta f_0 + \tfrac{1}{3} \Delta^2 f_0 + \right]$$

$$= y_0 + h \left[2y_0' + 2(E-1) y_0' + \tfrac{1}{3} (E^2 - 2E + 1) + y_0' \right]$$

neglecting higer order differences.

$$= y_0 + h \left[2y_0' + 2(y_1' - y_0') + \frac{1}{3} (y_2' - 2y_1' + y_0') \right]$$

Thus

$$y_2 = y_0 + \frac{h}{3} \left[y_0' + 4y_1' + y_2' \right]$$

Since x_0, x_1, x_2 are any three consecutive values of x of the above relation can be written as

$$y_{n+1,c} = y_{n-1} + \frac{h}{3} \left[y'_{n-1} + 4y'_n + y'_{n+1} \right] \quad \text{—③}$$

This is known as milne's corrector formula where the suffix c stands for corrector.

An improved value of y'_{n+1} is computed and again the corrector formula is applied until we get y_{n+1} to the desired accuracy.

10.7 Adams – Bashforth method

Consider $\frac{dy}{dx} = f(x, y)$ with $f(x_0) = y_0$. Newton's backward interpolation formula can be written as

$$f(x, y) = f_0 + n\nabla f_0 + \frac{n(n+1)}{2} \nabla^2 f_0 + \frac{n(n+1)(n+2)}{6} \nabla^3 f_0 + \cdots$$

Substituting this,

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\, dx \text{ , we get } \text{—①}$$

$$y_1 = y_0 + \int_{x_0}^{x_1} \left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2!} \nabla^2 f_0 + \cdots \right) dx$$

$$= y_0 + h \int_0^1 \left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2!} \nabla^2 f_0 + \cdots \right) dh$$

(putting $x = x_0 + nh$)

$$= y_0 + h \left(f_0 + \frac{1}{2} \nabla f_0 + \frac{5}{12} \nabla^2 f_0 + \frac{3}{8} \nabla^3 f_0 + \cdots \right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_0, \nabla^2 f_0, \nabla^3 f_0$ in terms of function values we get,

$$y_1 = y_0 + \frac{h}{24}\left[55 y_0' - 59 y_{-1}' + 37 y_{-2}' - 9 y_{-3}'\right]$$

This can also be written as

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55 y_n' - 59 y_{n-1}' + 37 y_{n-2}' - 9 y_{n-3}'\right]$$

This is called Adams - Bashforth predictor formula

A corrector formula can be derived in a similar manner by using Newton's backward difference formula at f_1.

ie) $f(x,y) = f_1 + n \nabla f_1 + \frac{n(n+1)}{2!} \nabla^2 f_1 + \frac{n(n+1)(n+2)}{3!} \nabla^3 f_1 + \cdots$

Substituting this in ① we get,

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] \text{ --- ②}$$

Solve $\frac{dy}{dx} = \frac{1}{x+y}$ $y(0) = 2$, $y(0.2) = 2.09$; $y(0.4) = 2.17$ $y(0.6) = 2.24$ find $y(0.8)$ using Milne's method

Soln: We are given that, $y_0 = 2, y_1 = 2.09$; $y_2 = 2.17$; $y_3 = 2.24$ and $h = 0.2$

The given differential equation is,

$$y' = \frac{1}{x+y} \text{ --- ③}$$

from the above equation we calculate y_1', y_2', y_3'

$$y_1' = \left(\frac{1}{x+y}\right)_{(x_1,y_1)} = \frac{1}{0.2+2.09} = 0.4367$$

$$y_2' = \left(\frac{1}{x+y}\right)_{(x_2,y_2)} = \frac{1}{0.4+2.17} = 0.3891$$

$$y_3' = \left(\frac{1}{x+y}\right)_{(x_3,y_3)} = \frac{1}{0.6+2.24} = 0.3521$$

Substituting these values in ② we get,

$$y_{4,p} = 2 + \frac{4 \times 0.2}{3}(2 \times 0.4367 - 0.3891 + 2 \times 0.3521)$$

$y_{4,p} = 2.3169$ --- ④ (correct to 4 decimal places)

Milne's correct formula is,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y_{n-1}' + 4y_n' + y_{n+1}'\right) \text{ --- ⑤}$$

putting $n=3$ is ⑤ we get,

$$y_{4,c} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right) \text{ --- ⑥}$$

Now, $y'_4 = \left(\frac{1}{x+y}\right)_{(x_4, y_4)} = \frac{1}{0.8+2.3169}$ (using ④)

∴ ⑥ becomes, $y_{4,c} = 2.17 + \frac{0.2}{3}(0.3891 + 4\times 0.3521 + 0.3208)$

$= 2.3112$ (correct to 4 decimal places)

Hence $y(0.8) = 2.3112$.

Pb Using Milne's predictor corrector method find $y(0.4)$ for the differential equation $\frac{dy}{dx} = 1+xy$, $y(0)=2$.

U.Q 5marks

$y(x_0) = y_0$

Taylor's method 8marks Katurvani

Soln:- Milne's predictor formula is.

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \text{ --- ①}$$

$x_0 = 0.$ $y_0 = 2$

Here $x_0 = 0$, $y_0 = 2$, $h = 0.1$

By Taylor's series method we have

$x_1 = 0.1$, $y_1 = 2.1103$ ($y(0.1)$)

$x_2 = 0.2$, $y_2 = 2.2430$ $y(0.2)$ $h = 0.1$

$x_3 = 0.3$, $y_3 = 2.4011$ $y(0.3)$ (Refer problem 1 in 10.1)

Now,

$(1+xy)$

$y'_1 = (y')_{(x_1, y_1)} = 1 + (0.1)(2.1103) = 1.21103$

$y'_2 = (y')(x_2, y_2) = 1 + (0.2)(2.243) = 1.4486$

$y'_3 = (y')(x_3, y_3) = 1 + (0.3) + (2.4011) = 1.72033$

putting these values in ① we get,

$$y_{4,p} = 2 + \frac{4(0.1)}{3}\left[2(1.21103) - 1.4486 + 2(1.72033)\right]$$

$= 2.5885$

Milne's correct formula is,

$$y_{4,c} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) \text{ --- ②}$$

Now,

$(1+xy)$

$y'_4 = (y')_{(x_4, y_4)} = 1 + (0.4)(2.5885)$

$= 2.0354$

∴ ② becomes, $y_{4,c} = 2.243 + \frac{0.1}{3}(1.4486 + 4(1.72033) + 2.0354)$

$= 2.5885$

Hence,

$y(0.4) = 2.5885$

pb Given $\frac{dy}{dx} = \frac{1}{2}(1+x^2)y^2$ and $y(0)=1$, $y(0.1)=1.06$, $y(0.2)=1.12$, $y(0.3)=1.21$, Evaluate $y(0.4)$ by Milne's predictor Corrector method.

proof:- Milne's predictor formula is

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

putting $n=3$ we get,

$$y_{4,P} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) \quad \text{--- (1)}$$

Here $x_0=0$, $y_0=1$, $x_1=0.1$, $y_1=1.06$

$x_2=0.2$, $y_2=1.12$, $x_3=0.3$, $y_3=1.21$

Now,

$$y'_1 = (y')_{(x_1,y_1)} = \frac{1}{2}(1+x_1^2)y_1^2$$

$$= \frac{1}{2}(1+(0.1)^2)(1.06)^2 = 0.5674$$

$$y'_2 = (y')_{(x_2,y_2)} = \frac{1}{2}(1+x_2^2)y_2^2$$

$$= \frac{1}{2}(1+(0.2)^2)(1.12)^2 = 0.6523$$

$$y'_3 = (y')_{(x_3,y_3)} = \frac{1}{2}(1+x_3^2)y_3^2$$

$$= \frac{1}{2}(1+0.3^2)(1.21)^2$$

$$= 0.7979$$

putting there values in (1) we get

$$y_{4,P} = 1 + \frac{4(0.1)}{3}\left[2(0.5674 - 0.6523 + 2(0.79791)\right]$$

$$= 1.2771$$

Milne's corrector formula is,

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

putting $n=3$ we get,

$$y_{4,C} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) \quad \text{--- (2)}$$

Now,

$$y'_4 = (y')(x_4, y_4) = \frac{1}{2}(1+x_4^2)y_4^2$$

$$= \frac{1}{2}(1+0.4^2)(1.2771)^2$$

$$= 0.9460$$

$\therefore$ ② becomes,

$$y_{4,c} = 1.12 + \frac{0.1}{3}[0.6523 + 4(0.7979) + 0.9460]$$

$$= 1.2797$$

$\therefore y(0.4) = 1.2797$

Now,

$$y_4' = (y')_{(x_4, y_4)} = \frac{1}{2}(1 + x_4^2)(y_4^2)$$

$$= \frac{1}{2}(1 + (0.4)^2)(1.2797)^2$$

$$= 0.9498$$

By applying Milne's corrector formula again

$$y_{4,c_1} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1.12 + \frac{0.1}{3}(0.6523 + 4(0.7979) + 0.9498]$$

$$= 1.2798$$

Now,

$$y_4' = (y')_{(x_4, y_4)} = \frac{1}{2}(1 + x_4^2)\, y_4^2$$

$$= \frac{1}{2}(1 + 0.4^2)(1.2798)^2$$

$$= 0.9500$$

By applying Milne's corrector formula,

$$y_{4,c_2} = 1.12 + \frac{0.1}{3}[0.6523 + 4(0.7979) + 0.9500]$$

$$= 1.2798$$

$y_{4,c_1} = y_{4,c_2} = 1.2798$ is the required soln

$\therefore y(0.4) = 1.2798$

Pb find $y(0.8)$ by Milne's method for the equation $y' = y - x^2$, $y(0) = 1$ obtaining the starting values by Taylor's series method.

Soln:- Given $y' = y - x^2$ —— ①

and $x_0 = 0$, $y_0 = 1$ and $h = 0.2$

first we find the starting values $y(0.2)$, $y(0.4)$ and $y(0.6)$ by Taylor's method.

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The Taylor's algorithm is,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \quad \text{—— (2)}$$

Differentiating (1) with respect to x we get,

$$y'' = y' - 2x$$

$$y''' = y'' - 2$$

$$\therefore y_0' = (y')_{(x_0, y_0)} = y_0 - x_0^2 = 1$$

$$y_0'' = (y'')_{(x_0, y_0)} = y_0 - x_0^2 = 1$$

$$y_0'' = (y'')_{(x_0, y_0)} = y_0' - 2x_0 = 1$$

$$y_0''' = (y''')_{(x_0, y_0)} = y_0'' - 2 = 1 - 2 = -1$$

using this in (2) we get,

$$y_1 = 1 + 0.2 + \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{6}(-1)$$

ie) $y(0.2) = 1.2187$

Now,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots \quad \text{—— (3)}$$

$$x_1 = x_0 + h = 0.2$$

$$(y_1') = (y')_{(x_1, y_1)} = y_1 - x^2 = 1.2187 - (0.2)^2 = 1.1787$$

$$(y_1'') = (y'')_{(x_1, y_1)} = y_1' - 2x_1 = 1.1787 - 0.4 = 0.7787$$

$$y_1''' = (y''')_{(x_1, y_1)} = y_1'' - 2 = -1.2213$$

using these values in (3) we get,

$$y_2 = 1.2187 + (0.2)(1.1787) + \frac{(0.2)^2}{2}(0.7787) + \frac{(0.2)^3}{6}(-1.2213)$$

ie) $y(0.4) = 1.4684$

Now, $y(0.6) = y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \quad \text{—— (4)}$

$$y_2' = (y')_{(x_2, y_2)} = y_2 - x_2^2 = 1.4684 - (0.4)^2$$

$$= 1.3084$$

$$y_2'' = (y'')_{(x_2, y_2)} = y_2' - 2x_2 = 1.3084 - 0.8$$

$$= 0.5084$$

$$y_2''' = y''_{(x_2, y_2)} = y_2'' - 2 = 0.5884 - 2$$

$$= -1.4916$$

using these values (1) & (4) we get

$$y_3 = 1.468 + (0.2)(1.3084) + \frac{(0.2)^2}{2}(0.5084) + \frac{(0.2)^3}{6}(-1.4916)$$

i.e) $y(0.6) = 1.7383$

Milne's predictor formula is,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

putting $n=3$ we get,

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right]$$

Now,

$$y'_3 = (y')_{(x_3,y_3)} = y_3 - x_3^2 = 1.7383 - (0.6)^3$$
$$= 1.5223$$

$$\therefore y_{4,p} = 1 + \frac{4(0.2)}{3}\left(2(1.1787) - 1.3084 + 2(1.5223)\right)$$
$$= 2.0916.$$

Milne's corrector formula is,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

putting $n=3$, we get,

$$y_{4,c} = \cancel{y_2} + \frac{h}{3}\left[\cancel{y'_{n-1}} + \cancel{4y'_n} + \cancel{y'_{n+1}}\right]$$

$$y_{4,c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

Now,

$$y'_4 = (y')_{(x_4,y_4)} = y_4 - x_4^2$$
$$= 2.0916 - (0.8)^2$$
$$= 1.4516$$

$$\therefore y_{4,c} = 1.4684 + \frac{0.2}{3}\left[1.3084 + 4(1.5223) + 1.4516\right]$$
$$= 2.0583$$

$\therefore y(0.8) = 2.0583$

Pb using Adam's Bashforth method find $y(4.4)$ given $5xy' + y^2 = 2$ $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$

soln: Given $y' = \frac{2-y^2}{5x}$. let $h = 0.1$

$x_0 = 4$, $y_0 = 1$, $x_1 = 4.1$, $y_1 = 1.0049$

$x_2 = 4.2$, $y_2 = 1.0097$, $x_3 = 4.3$, $y_3 = 1.0143$

Adam's predictor formula is,

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right]$$

putting $n=3$ we have,

$y_{4+P} = y_3 + \frac{h}{24}(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0)$ ——①

$y'_0 = (y')_{(x_0,y_0)} = \frac{2-y_0^2}{5x_0} = 0.05$

$y'_1 = (y')_{(x_1,y_1)} = \frac{2-y_1^2}{5x_1} = 0.0483$

$y'_2 = (y')_{(x_2,y_2)} = \frac{2-y_2^2}{5x_2} = 0.0467$

$y'_3 = (y')_{(x_3,y_3)} = \frac{2-y_3^2}{5x_3} = 0.0452$

using the values in ① we get,

$y_{4,P} = 1.0143 + \frac{0.1}{24}[55(0.0452) - 59(0.0467) + 37(0.0483) - 9(0.05)]$

$= 1.01413 + \frac{0.1}{24}(4.2731 - 3.2053)$

$= 1.0186$

$\therefore y(4.4) = 1.0186$

Adam's Corrector formula is

$y_{n+1,c} = y_n + \frac{h}{24}(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2})$

putting $n=3$ we get,

$y_{4,c} = y_3 + \frac{h}{24}[9y'_4 + 19y'_3 - 5y'_2 + y'_1]$ ——②

Now, $y'_4 = (y')_{(x_4,y_4)} = \frac{2-y_4^2}{5x_4} = 0.0437$

$\therefore$ ② becomes,

$y_{4,c} = 1.0143 + \frac{0.1}{24}[9(0.0437) + 19(0.0452) - 5(0.0467) + (0.0483)]$

$= 1.0143 + \frac{0.1}{24} \times 1.0669$

$y(4.4) = 1.0187$

Pb using Adams Bashforth method, determine $y(1.4)$ given that $y' - x^2y = x^2$, $y(1) = 1$. obtain the starting values from Euler's method

Soln:- The Euler's algorithm for the differential equation $\frac{dy}{dx} = f(x,y)$ is given by

$y_{n+1} = y_n + hf(x_n, y_n)$, $n = 0,1,2,3$ ——①

Here $f(x,y) = x^2(1+y)$, $x_0 = 1$, $y_0 = 1$ and take $h = 0.1$

putting $n=0$ in ① we get,

$y_1 = y_0 + h f(x_0, y_0) = 1 + (0.1)(2) = 1.2$

putting $n=1$ in ① we get,

$y_2 = y_1 + h f(x_1, y_1) = y_1 + h[x_1^2 (1+y_1)]$

$= 1.2 + (0.1)[(1.1)^2 \times (2.2)]$

$= 1.2 + (0.1)(2.662)$

$= 1.4662$

putting $n=2$ in ① we get,

$y_3 = y_2 + h f(x_2, y_2) = 1.4662 + 0.1 [(1.2)^2 \times 2.4662]$

$= 1.8213$

Adam's predictor formula is,

$$y_{n+1,P} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

putting $n=3$ we get,

$$y_{4,P} = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0] \quad \text{②}$$

Now,

$y'_0 = [x^2(1+y)]_{(x_0,y_0)} = 2$

$y'_1 = [x^2(1+y)]_{(x_1,y_1)} = (1.1)^2(1+1.2) = 2.662$

$y'_2 = [x^2(1+y)]_{(x_2,y_2)} = (1.2)^2(1+1.4662)$

$= 3.5513$

$y'_3 = [x^2(1+y)]_{(x_3,y_3)} = (1.3)^2(1+1.8213)$

$= 4.7680$

Putting these values in ② we get,

$y_{4,P} = 1.8213 + \frac{0.1}{24}[55(4.768) - 59(3.5513) + 37(2.662) - 9(2)]$

$\therefore y(1.4) = 2.3763$ (by predictor formula)

Adam's corrector formula is

$$y_{n+1,c} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

putting $n=3$ we get,

$$y_{4,c} = y_3 + \frac{h}{24}[9y'_4 + 19y'_3 - 5y'_2 + y'_1] \quad \text{③}$$

Now,

$y'_4 = [x^2(1+y)]_{(x_4,y_4)} = (1.4)^2(1+2.3763)$

$= 6.6175$

$\therefore$ ③ becomes,

$$y_{4,C} = 1.8213 + \frac{0.1}{24}\left[9(6.6175) + 19(4.768) - 5(3.5513) + 2.662\right]$$

$$= 1.8213 + \frac{0.1}{24}(135.055)$$

$$= 2.3840$$

$\therefore y(1.4) = 2.3840$ //.

7) Pb using Adam's Bashforth method find $y(0.4)$ given that $y' = 1+xy$, $y(0)=2$.

Soln:-

Adam's predictor formula for $n=3$ is

$$y_{4,P} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right] \quad \text{— ①}$$

Here $x_0 = 0$, $y_0 = 2$ Take $h = 0.1$

By Taylor's series method we have $x_1 = 0.1$,

$y(0.1) = y_1 = 2.1103$,

$x_2 = 0.2$, $y_2 = 2.243$, $x_3 = 0.3$ and $y_3 = 2.4011$

(Refer pro ① in 10.1)

Now, $y_0' = y'(x_0, y_0) = 1$

|||ly $y_1' = 1.21103$, $y_2' = 1.4486$ and $y_3' = 1.72033$

using there values in ① we get,

$$y_{4,p} = 2.4011 + \frac{0.1}{24}\left[55(1.72033) - 59(1.4486) + 37(1.21103) - 9\right]$$

$$= 2.5884.$$

Adam's corrector formula is,

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right] \quad \text{--- (2)}$$

Now,

$$\cancel{y_4' = 1 + (0.4)(2.5885)}$$

$$\cancel{= 2.0354}$$

Now,

$$y_4' = (y')_{(x_4, y_4)} \cancel{= 1 + (0.4)(}$$

Now,

$$y_4' = (y')_{(x_4, y_4)} = 1 + (0.4)(2.5885)$$

$$= 2.0354$$

$\therefore$ (2) becomes,

$$y_{4,c} = 2.4011 + \frac{0.1}{24}\left[9(2.0354) + 19(1.72033) - 5(1.4486) + 1.21103\right]$$

$$= 2.5885$$

Hence

$$y(0.4) = 2.5885$$